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A NOTE ON QUASICENTRAL APPROXIMATE UNITS IN $B(H)$

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ABSTRACT. If a Hilbert Space, H, is infinite dimensional, $B(H)$ has no countable quasicentral approximate unit for the ideal of finite rank operators.

Quasicentral approximate units were introduced in [2] by W. Arveson and independently by C. Akemann and G. K. Pedersen in [1]. Arveson has shown that for any C^*-algebra, A, quasicentral approximate units exist for all ideals of A. Further, if A is a separable C^*-algebra, the quasicentral approximate unit may be taken to be an increasing sequence. While countable quasicentral approximate units can sometimes be found in the inseparable case, Arveson observes in [3] that it seems unlikely there is such an approximate unit for the ideal of finite rank operators in $B(H)$. Here we exhibit a proof of this fact.

Definition. Given a C^*-algebra, A, and an ideal, K, of A, an increasing net $\{u_\lambda\}$ of positive elements of A is called an approximate unit for K if $\|u_\lambda\| \leq 1$ and $\lim_\lambda \|u_\lambda k - k\| = 0$ for all k in K. An approximate unit is said, further, to be quasicentral if, for any a in A, $\lim_\lambda \|u_\lambda a - au_\lambda\| = 0$.

Remarks. Recall that if $\{u_\lambda\}$ is an approximate unit for the ideal of finite rank operators in $B(H)$ then $\{u_\lambda\}$ must converge to the identity in the strong operator topology since $u_\lambda R - R$ converges to zero for each rank one operator R. Thus the following proposition suffices to show that $B(H)$ has no countable quasicentral approximate unit for the ideal of finite rank operators.

Throughout, we shall denote by $a \otimes b$ the rank one operator which maps x to $\langle x, a\rangle b$ and assume that H is an infinite dimensional Hilbert space.

Proposition. *Let (F_n) be an increasing sequence of positive finite rank operators in $B(H)$ and suppose (F_n) converges to the identity, I, in the strong operator topology. Then there is a partial isometry, U, such that $F_n U - UF_n$ does not converge to zero in norm.*

Proof. Since $0 \leq F_1 \leq F_2 \leq \cdots$, it follows that their ranges form an increasing sequence of subspaces. We can find an orthonormal sequence ϕ_k

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and a sequence of integers, k_n, such that the range of each F_n is the span $\{\phi_1, \dots, \phi_{k_n}\}$. Clearly for each i, $\lim\langle F_n\phi_i, \phi_i\rangle = 1$.

Now choose an integer n_1 such that $\langle F_{n_1}\phi_1, \phi_1\rangle \geq \frac{1}{2}$ and let $U_1 = \phi_1 \otimes \phi_{k_{n_1}+1}$. Inductively obtain an increasing sequence, n_i, such that $\langle F_{n_i}\phi_i, \phi_i\rangle \geq \frac{1}{2}$ and $k_{n_i} > k_{n_{i-1}}$. Let $U_i = \phi_i \otimes \phi_{k_{n_i}+1}$ for each i. Since this forms a sequence of partial isometries with orthogonal intitial and final spaces, $U = \sum U_i$ is a partial isometry and $F_{n_i}U\phi_i = 0$ for all i, since $U\phi_i$ is orthogonal to the range of the selfadjoint operator, F_{n_i}. Our result now follows from the inequalities below.

$$\begin{aligned}\|UF_{n_i} - F_{n_i}U\| &\geq |\langle (UF_{n_i} - F_{n_i}U)\phi_i, \phi_{k_{n_i+1}}\rangle| \\ &= |\langle UF_{n_i}\phi_i, \phi_{k_{n_i+1}}\rangle| \\ &= |\langle F_{n_i}\phi_i, \phi_i\rangle| \geq \tfrac{1}{2} \quad \text{for each } i.\end{aligned}$$

References

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