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ON THE GLAUBERMAN CORRESPONDENCE

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ABSTRACT. In this paper we give an elementary proof of the p-group case of Glauberman's correspondence.

1. INTRODUCTION

Let p be a prime number.

Let S be a p-group acting on a finite p'-group G. Let $C = C_G(S)$ and $\mathrm{Irr}_S(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi^s = \chi\,,\ \forall s \in S\}$.

It is well known that the map $K \to K \cap C$ defines a bijection from the set of S-invariant conjugacy classes of G onto the set of conjugacy classes of C [2, 13.10].

Let R be the full ring of algebraic integers in $\mathbf{C}$, let M be a maximal ideal of R containing pR, and set $F = R/M$. Let $^*: R \to F$ the canonical homomorphism.

For $\chi \in \mathrm{Irr}(G)$, the map defined on $Z(F[G])$ by $\lambda_\chi(\widehat{K}) = \omega_\chi(\widehat{K})^*$ is an algebra homomorphism from $Z(F[G])$ to F. Note that $\lambda_\chi(\widehat{K}) = (\chi(x)|K|/\chi(1))^*$ for $x \in K$.

2. TWO LEMMAS

2.1. Lemma. *Let $\chi \in \mathrm{Irr}_S(G)$. We define $\delta_\chi: Z(F[C]) \to F$ by setting $\delta_\chi(\widehat{K \cap C}) = (\chi(x)|K|/\chi(1))^*$ for K S-invariant conjugacy class of G and $x \in K \cap C$. Then δ_χ is an algebra homomorphism.*

Proof. Since $\delta_\chi(\widehat{K \cap C}) = \lambda_\chi(\widehat{K})$, it suffices to show that

$$\delta_\chi(\widehat{K_i \cap C}\widehat{K_j \cap C}) = \lambda_\chi(\widehat{K}_i\widehat{K}_j)$$

for K_i, K_j, S-invariant conjugacy classes of G.

Write $K_1, \ldots, K_h$ for the S-invariant conjugacy classes, and $K_{h+1,1}, \ldots, K_{h+1,a_1}, \ldots, K_{h+t,1}, \ldots, K_{h+t,a_t}$ for the rest, where $K_{h+j,1}, \ldots, K_{h+j,a_j}$ is an S- orbit. Note that $a_j = p^{b_j} > 1$.

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It is clear that we can write

$$\widehat{K}_i\widehat{K}_j = \sum_{k=1,\dots,h} a_{ijk}K_k + \sum_{l=1,\dots,t} b_{ijl}\Big(\sum_{m=1,\dots,a_l} \widehat{K}_{h+l,m}\Big).$$

Fix $x_k \in K_k \cap C$. Since $a_{ijk} = |\{(x,y) \in K_i \times K_j : xy = x_k\}|^*$ and $|\{(x,y) \in K_i \times K_j : xy = x_k\}| \equiv |\{(x,y) \in K_i \cap C \times K_j \cap C : xy = x_k\}| \bmod p$, we have that $\widehat{K_i \cap C}\widehat{K_j \cap C} = \sum_{k=1,\dots,h} a_{ijk}\widehat{K_k \cap C}$.

Now, since λ_χ is constant over each S-orbit, we have

$$\begin{aligned}\lambda_\chi(\widehat{K}_i\widehat{K}_j) &= \sum_{k=1,\dots,h} a_{ijk}\lambda_\chi(\widehat{K}_k) + \sum_{l=1,\dots,t} b_{ijl}p^{b_l}\lambda_\chi(\widehat{K}_{h+l,1}) \\ &= \sum_{k=1,\dots,h} a_{ijk}\lambda_\chi(\widehat{K}_k) \\ &= \sum_{k=1,\dots,h} a_{ijk}\delta_\chi(\widehat{K_k \cap C}) \\ &= \delta_\chi\Big(\sum_{k=1,\dots,h} a_{ijk}\widehat{K_k \cap C}\Big) = \delta_\chi(\widehat{K_i \cap C}\widehat{K_j \cap C}).\end{aligned}$$

The following result is well known. We give the proof to make it clear that no results on p-blocks are needed for this paper.

2.2. Lemma. *Suppose p does not divide $|G|$. The maps λ_χ for $\chi \in \mathrm{Irr}(G)$ are distinct and are all the algebra homomorphisms from $Z(F[G])$ to F.*

Proof. $F[G]$ is a direct sum of full matrix rings over F, so $Z(F[G]) \cong F^k$ where $k = \mathrm{cl}(G)|$. There are thus k algebra homomorphisms $Z(F[G]) \to F$ and it is enough to show that the λ_χ are distinct.

Let $e_\chi = \chi(1)\sum_{g\in G}\chi(g^{-1})^* g \in Z(F[G])$. Then $\lambda_\chi(e_\chi) = |G|^* \neq 0$ and $\lambda_\psi(e_\chi) = 0$ for $\chi \neq \psi$. The result now follows.

3. The Glauberman correspondence

Notation. Given $\chi \in \mathrm{Irr}_S(G)$, since δ_χ is an algebra homomorphism $Z(F[C]) \to F$, it follows that there exists a unique $\widetilde{\chi} \in \mathrm{Irr}(C)$ such that $\delta_\chi = \lambda_{\widetilde{\chi}}$. Thus for $x \in C$, we have

$$(\chi(x)|K|/\chi(1))^* = (\widetilde{\chi}(x)|K \cap C|/\widetilde{\chi}(1))^*, \quad \text{where } K = Cl_G(x).$$

Since $|K| \equiv |K \cap C| \not\equiv 0 \bmod p$, this gives $\widetilde{\chi}(1)\chi(x) \equiv \chi(1)\widetilde{\chi}(x) \bmod M$, for all $x \in C$.

3.1. Theorem. *The map $\mathrm{Irr}_S(G) \to \mathrm{Irr}(C)$ defined by $\chi \to \widetilde{\chi}$ is a bijection. Also, $[\chi_C, \widetilde{\chi}] \equiv \pm 1 \bmod p$, and $[\chi_C, \theta] \equiv 0 \bmod p$ for $\widetilde{\chi} \neq \theta \in \mathrm{Irr}(C)$.*

Proof. Let $\chi \in \mathrm{Irr}_S(G)$ and $\theta \in \mathrm{Irr}(C)$. Then

$$\begin{aligned}|C|\widetilde{\chi}(1)[\chi_C, \theta] = \widetilde{\chi}(1)\sum_{x\in C}\chi(x)\theta(x^{-1}) &\equiv \chi(1)\sum_{x\in C}\widetilde{\chi}(x)\theta(x^{-1}) \\ &= \chi(1)[\widetilde{\chi}, \theta]|C| \quad \bmod M.\end{aligned}$$

Since p does not divide $|C|$, this gives $\tilde{\chi}(1)[\chi_C, \theta] \equiv \chi(1)[\tilde{\chi}, \theta] \bmod p$. Since p does not divide $\tilde{\chi}(1)$, taking $\theta \neq \tilde{\chi}$, this gives $[\chi_C, \theta] \equiv 0 \bmod p$, and taking $\theta = \tilde{\chi}$ gives

$$\tilde{\chi}(1)[\chi_C, \tilde{\chi}] \equiv \chi(1) \not\equiv 0 \bmod p. \tag{1}$$

Thus $\tilde{\chi}$ is the unique irreducible consituent of χ_C with multiplicity not divisible by p.

Now let $\chi, \varphi \in \mathrm{Irr}_S(G)$. Then

$$|G|[\chi, \varphi] = \sum_{x \in G} \chi(x)\varphi(x^{-1}) \equiv \sum_{x \in C} \chi(x)\varphi(x^{-1}) = |C|[\chi_C, \varphi_C] \quad \bmod M$$

(using that χ, φ are S-invariant). Also, $|G| \equiv |C| \not\equiv 0 \bmod p$, and so $[\chi, \varphi] \equiv [\chi_C, \varphi_C] \bmod p$.

Since $\chi_C = [\chi_C, \tilde{\chi}]\tilde{\chi} + p\Delta$ and $\varphi_C = [\varphi_C, \tilde{\varphi}]\tilde{\varphi} + pE$, we have that $[\chi, \varphi] = [\chi_C, \tilde{\chi}][\varphi_C, \tilde{\varphi}][\tilde{\chi}, \tilde{\varphi}] \quad \bmod p$. (2)

This shows that our map is injective.

If $\alpha \in \mathrm{Irr}(C)$ is not in the image of our map, then $[\chi_C, \alpha] = [\chi, \alpha^G] \equiv 0 \bmod p$ for all $\chi \in \mathrm{Irr}_S(G)$. Since α^G is S-invariant, $[\alpha^G, \varphi^s] = [\alpha^G, \varphi]$ for all $\varphi \in \mathrm{Irr}(G)$, for all $s \in S$. This implies that p divides $\alpha^G(1)$, a contradiction.

Finally, taking $\chi = \varphi$ in (2), we get $1 \equiv [\chi_C, \tilde{\chi}]^2 \bmod p$ and so $[\chi_C, \tilde{\chi}] \equiv \pm 1 \bmod p$ as claimed.

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