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MUTATION OF KNOTS

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ABSTRACT. In general, mutation does not preserve the Alexander module or the concordance class of a knot.

For a discussion of mutation of classical links, and the invariants which it is known to preserve, the reader is referred to [LM, APR, MT]. Suffice it here to say that mutation of knots preserves the polynomials of Alexander, Jones, and Homfly, and also the signature. *Mutation* of an oriented link k can be described as follows. Take a diagram of k and a tangle T with two outputs and two inputs, as in Figure 1.

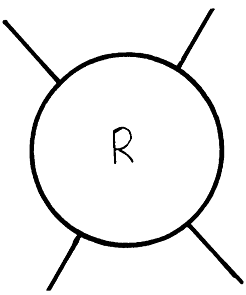

FIGURE 1

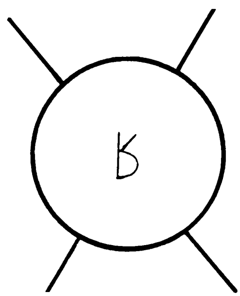

FIGURE 2

Rotate the tangle about the east-west axis to obtain Figure 2, or about the north-south axis to obtain Figure 3, or about the axis perpendicular to the paper to obtain Figure 4. Keep or reverse all the orientations of T as dictated by the rest of k. Each of the links so obtained is a *mutant* of k.

The *reverse* k' of a link k is obtained by reversing the orientation of each component of k. Let us adopt the convention that a *knot* is a link of one component, and that $k + l$ denotes the connected sum of two knots k and l.

Lemma. *For any knot k, the knot $k + k'$ is a mutant of $k + k$.*

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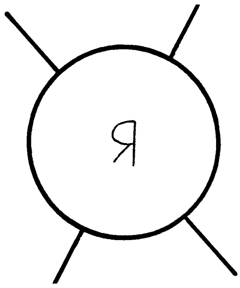

FIGURE 3

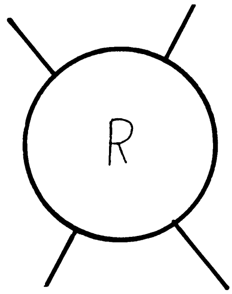

FIGURE 4

Proof. Shrink one of the summands in $k + k$ to a small knot, and arrange a diagram of $k + k$ to have a tangle as in Figure 5. Rotate about the axis perpendicular to the page, to obtain Figure 6, which represents $k + k'$. Note that whatever convention we make about orientations, we always obtain $k + k'$. Q.E.D.

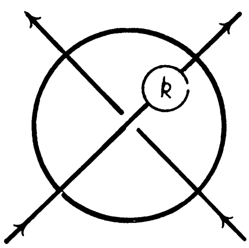

FIGURE 5

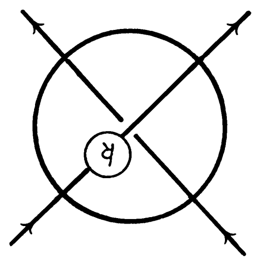

FIGURE 6

By a result of C. Livingston [L], there exist knots k which are not concordant to their reverses k'. It follows at once that $k+k$ is not concordant to $k+k'$, and hence that mutation does not preserve the concordance class in general. I should like to thank Cameron Gordon for reminding me of Livingston's result.

In [K] there is an example of a knot k, in fact the pretzel knot $(25, -3, 13)$, whose Steinitz-Fox-Smythe row ideal class ρ does not satisfy $\rho^2 = 1$. The row ideal class of k', as pointed out in [K], is τ, the column ideal class of k. Of course, $\rho\tau = 1$, and so we see that the row ideal class of $k+k$ is $\rho^2 \neq 1$, whereas the row ideal class of $k+k'$ is $\rho\tau = 1$. Thus we have an example in which the knot module of $k+k$ is not isomorphic to that of $k+k'$. Another example can be obtained from [BHK, §4], and other examples can be found using [B] and number theory tables.

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