PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 1, January 1989

THREE-SPACE PROBLEMS FOR THE APPROXIMATION PROPERTIES

GILLES GODEFROY AND PIERRE DAVID SAPHAR

(Communicated by William J. Davis)

ABSTRACT. Let M be a closed subspace of a Banach space X. We suppose that M has the B.A.P. and that $M^{\perp}$ is complemented in X^*. Then, if X/M has the B.A.P. (resp. the A.P.), the space X has the same property. There are similar results if M is an $\mathscr{L}_\infty$ space. If X/M is an $\mathscr{L}_1$ space, then X has the B.A.P. if and only if M has the B.A.P. We notice that the quotient algebra $L(H)/K(H)$ (H infinite-dimensional Hilbert space) does not have the A.P.

1. INTRODUCTION

Let X be a Banach space, and M a closed subspace of X. Assume that the spaces M and X/M have the bounded approximation property (B.A.P.); what can be said about X? It is known that this does not imply in general that X the approximation property (A.P.); Indeed W. B. Johnson and H. P. Rosenthal have shown in [6] that every separable space X contains a subspace M such that both M and X/M have a finite-dimensional decomposition. More recently, W. Lusky in [12] has shown that if X is separable and contains a subspace isomorphic to c_0, then there exists a subspace M of X with a basis such that X/M has a shrinking basis. However, some positive results can be obtained under simple additional assumptions.

A typical result is the following: If M is a closed subspace of a Banach space X such that $M^{\perp}$ is complemented in X^*, and if X/M has the B.A.P., then X has the B.A.P. if and only if M has the B.A.P. We also show that if M is an $\mathscr{L}_\infty$ space and X/M has the A.P. (resp. the B.A.P.), then X has the A.P. (resp. the B.A.P.). On the other hand, if X/M is an $\mathscr{L}_1$ space, X has the B.A.P. if and only if M has the B.A.P. We deduce from Szankowski's result [13], that the quotient algebra $L(H)/K(H)$ (H infinite-dimensional Hilbert space) does not have the A.P.

Notations. The space of bounded operators of a Banach space X is denoted by $L(X)$, and the space of finite rank operators by $R(X)$. For two Banach spaces

Received by the editors December 1, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46B20.
Key words and phrases. approximation property, three-space problem, extension of finite rank operators.

X and Y, the tensor product $X \otimes Y$ endowed with the projective norm π and completed will be denoted $X \otimes_\pi Y$. If X^* is the dual of X, the w^*-topology on $L(X^*)$ is the topology of pointwise convergence on the canonical predual $X^* \otimes_\pi X$ of $L(X^*)$. The topology w^*_{op} on $L(X^*)$ is the topology of pointwise convergence on the algebraic tensor product $X^* \otimes X$. Our reference on the approximation properties (originally defined in [4]) is [11, Section 1.e]. The $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces are defined and studied in [10].

2. Results

Our first lemma is a classical perturbation argument (see [2]).

Lemma 2.1. *Let X be a Banach space. Then:*

(1) *X has the A.P. if and only if Id_{X^*} belongs to the closure of $R(X^*)$ in $(L(X^*), w^*)$;*

(2) *X has the B.A.P. if and only if there exists $\lambda > 0$ such that Id_{X^*} belongs to the closure of $\{R; R \in R(X^*),\ \|R\| \leq \lambda\}$ in $(L(X^*), w^*_{\text{op}})$ (or in $(L(X^*), w^*)$).*

Proof. (1) Assume that X has the A.P. Let (R_α) be a net of finite rank operators from X into X such that $R_\alpha \to \text{Id}_X$ for the topology τ_k of compact convergence. By [11, Proposition 1.e.3] one has $\phi(R_\alpha) \to \phi(\text{Id}_X)$ for every $\phi \in X^* \otimes_\pi X$; hence $R^*_\alpha \xrightarrow{w^*} \text{Id}_{X^*}$.

Assume conversely that Id_{X^*} belongs to the closure of $R(X^*)$ in $(L(X^*), w^*)$. Let $U \in R(X^*)$. One verifies easily that there exists a net (T_α) in $R(X)$ such that $T^*_\alpha \xrightarrow{w^*} U$. Thus, there exists a net (V_α) in $R(X)$ such that $V^*_\alpha \xrightarrow{w^*} \text{Id}_{X^*}$. Hence, V_α converges to Id_X for the weak topology of $(L(X), \tau_k)$ and a convex combination argument shows that X has the A.P.

(2) If X has the B.A.P., it is clear that there exists $\lambda > 0$ such that Id_{X^*} belongs to the closure of $\{R; R \in R(X^*),\ \|R\| \leq \lambda\}$ in $(L(X^*), w^*)$. The converse is exactly Theorem 1 of [2].

Observe finally that by compactness, the topologies w^* and w^*_{op} coincide on the bounded subsets of $L(X^*)$. □

The next results will show the main tools for obtaining positive results in the "three-space" situation.

Lemma 2.2. *Let X be a Banach space, and M a closed subspace of X such that X/M has the A.P. If there exists a bounded net (T_α) in $R(X)$ such that*

$$\langle T_\alpha(x), x^* \rangle \to \langle x, x^* \rangle \quad \textit{for each } x \in M \textit{ and each } x^* \in X^*,$$

then X has the A.P.

Proof. The net (T^*_α) is a bounded net in the dual space $L(X^*)$. Let U be a w^*-cluster point of (T^*_α). Clearly $\langle x, U(x^*) \rangle = \langle x, x^* \rangle$ for each $x \in M$ and each $x^* \in X^*$. Then, if j is the canonical map from $M^\perp$ to X^*, there exists an operator D from X^* to $M^\perp$ such that $U - \text{Id}_{X^*} = jD$.

By assumption, X/M has the A.P., hence there is a net (S_β) in $R(X/M)$ such that (S_β^*) satisfies $S_\beta^* \xrightarrow{w^*} \mathrm{Id}_{M^\perp}$ in $L(M^\perp)$. If we let $V_\beta = jS_\beta^* D$, we have $V_\beta \xrightarrow{w^*} jD$ in $L(X^*)$. This shows that $\mathrm{Id}_{X^*} = U - jD$ belongs to the w^*-closure of the set $(T_\alpha^* - V_\beta)$; hence by 2.1(1), X has the A.P. □

In the case where X/M is assumed to have the B.A.P., we can state

Lemma 2.3. *Let X be a Banach space and M a closed subspace of X such that X/M has the B.A.P. Then the following are equivalent:*

(1) *X has the B.A.P.;*

(2) *There exists a bounded net (T_α) in $R(X)$ such that*

$$\forall x \in M, \forall x^* \in X^*, \langle T_\alpha(x), x^* \rangle \to \langle x, x^* \rangle.$$

Proof. (1) $\Rightarrow$ (2) is clear by restriction.

(2) $\Rightarrow$ (1). We repeat the proof of 2.2 with the same notation. Since X/M has the B.A.P., the net (S_β) may be taken bounded; then (V_β) is bounded and Id_{X^*} is in the w^*-closure of a bounded subset of $R(X^*)$. We conclude by 2.1(2). □

Let us now state the main result of this note.

Theorem 2.4. *Let X be a Banach space, and M a closed subspace of X such that $M^\perp$ is complemented in X^*. Then we have:*

(1) *If X has the A.P. (resp. the B.A.P.), M has the A.P. (resp. the B.A.P.);*

(2) *If M has the B.A.P., then*

X/M has the A.P. implies that X has the A.P.,

X/M has the B.A.P. implies that X has the B.A.P.

Proof. Let i be the canonical map from M to X. Since $M^\perp$ is complemented in X^*, there exists an operator σ from M^* to X^* such that $i^*\sigma = \mathrm{Id}_{M^*}$.

(1) Let (T_α) be a net in $R(X^*)$ such that $T_\alpha \xrightarrow{w^*} \mathrm{Id}_{X^*}$ in $L(X^*)$. We consider the operators $W_\alpha = i^* T_\alpha \sigma$; it is clear that $W_\alpha \in R(M^*)$ and that $W_\alpha \xrightarrow{w^*} \mathrm{Id}_{M^*}$ in $L(M^*)$. Moreover, $\|W_\alpha\| \le \|T_\alpha\| \, \|\sigma\|$. Hence, the net (W_α) is bounded if (T_α) is bounded. Lemma 2.1 concludes the proof.

(2) Let (R_α) be a bounded net in $R(M)$ such that $R_\alpha m \to m$, for every $m \in M$. Each operator (R_α) can be written

$$R_\alpha = \sum_{t=1}^{n(\alpha)} m_{t,\alpha}^* \otimes m_{t,\alpha}, \qquad m_{t,\alpha}^* \in M^*, m_{t,\alpha} \in M.$$

We define $S_\alpha \in R(X)$ by

$$S_\alpha = \sum_{t=1}^{n(\alpha)} \sigma(m_{t,\alpha}^*) \otimes m_{t,\alpha}.$$

It is clear that for every $m \in M$ and every $x^* \in X^*$

$$\langle S_\alpha(m), x^* \rangle = \langle R_\alpha(m), x^* \rangle \to \langle m, x^* \rangle.$$

Moreover, $\|S_\alpha\| \le \|\sigma\| \cdot \|R_\alpha\|$, and the net (S_α) is bounded. Now Lemmas 2.2 and 2.3 conclude the proof. □

We describe now a few consequences of this result. Our first observation deals with subspaces of X containing M.

Corollary 2.5. *Let M and Y be two subspaces of X such that $M \subset Y \subset X$. Suppose $M^\perp$ is complemented in X^*. Then, the orthogonal of M in Y^* is complemented in Y^*. Hence if M does not have the A.P. (resp. the B.A.P.), no space Y between M and X has the A.P. (resp. the B.A.P.).*

Proof. If we write $X^* = M^\perp \oplus Z$ then we have

$$Y^* = X^*/Y^\perp = (M^\perp/Y^\perp) \oplus Z$$

and the space $M^\perp/Y^\perp$ is precisely the orthogonal of M in Y^*. The conclusion follows by 2.4(1). □

Example 2.6. Let X be a Banach space, and $G = X^U$ an ultrapower of X (see, for instance [1 or 5]). If $x = \widehat{(x_i)}$ is an element of G, we can define a map σ from X^* to G^* by $\langle x, \sigma(f) \rangle = \lim_U(\langle x_i, f \rangle)$ for each f of X^*. It is clear that σ is a right inverse of the canonical map from G^* to X^*. Then $X^\perp$ is complemented in G^*. Hence, Theorem 2.4 applies to this situation. Let F be a subspace of G such that $X \subset F \subset G$; by Corollary 2.5 we obtain that if X does not have the A.P. (resp. the B.A.P.), it is the same for F. For a similar connection between finite representability and extensions, see [8].

The above applies for instance to any Banach space F such that $X \subset F \subset X^{**}$. In the case $F = X^{**}$, we can deduce from [7] more precise results, namely:

Corollary 2.7. *Let X be a Banach space. Let us call* (P) *one of the properties*:

(i) *Y has a basis*;
(ii) *Y has an F.D.D.*;
(iii) *Y is a π-space* (*see* [7, *p.* 489]);
(iv) *Y has the B.A.P.*

Then if X and X^{**}/X have (P), *X^{**} and X^* have* (P).

Proof. If X and X^{**}/X have the B.A.P., then X^{**} has the B.A.P. by 2.4(2), and thus X^* has the B.A.P. [11, Theorem 1.e.7] and (iv) is proved. Now the conclusion follows:

(a) if (P) is (i), from [7, Theorem 1.4.(b)];
(b) if (P) is (ii), from [7, Theorem 1.3)];
(c) if (P) is (iii), from [7, Corollary 4.8).] □

The next observation is a consequence of an important result of Szankowski (see [13]).

Corollary 2.8. *Let H be an infinite-dimensional Hilbert space, and $K(H)$ be the space of compact operators on H. Then the quotient algebra $L(H)/K(H)$ does not have the A.P.*

Proof. Since $L(H) = K(H)^{**}$, $K(H)^{\perp}$ is complemented in $L(H)^*$. The space $K(H)$ has the B.A.P. On the other hand, by [13], $L(H)$ does not have the A.P. Therefore, 2.4(2) concludes the proof. □

Let us finally show

Corollary 2.9. *Let M be a closed subspace of the Banach space X. Then:*

(1) *If M is an $\mathscr{L}_\infty$ space and X/M has the A.P. (resp. the B.A.P.), then X has the A.P. (resp. the B.A.P.);*

(2) *If X/M is an $\mathscr{L}_1$ space, then X has the B.A.P. if and only if M has the B.A.P.*

Proof. (1) If M is an $\mathscr{L}_\infty$ space (see [10]), then M has the B.A.P. Moreover, there exists a constant K such that every finite rank operator $R: M \to M$ admits an extension $\tilde{R}: X \to M$ of finite rank, with $\|\tilde{R}\| \leq K. \|R\|$. Hence there exists a bounded net (T_α) in $R(X)$ such that $(T_\alpha(x))$ converges weakly to x for every $x \in M$. Lemmas 2.2 and 2.3 conclude the proof.

(2) If X/M is an $\mathscr{L}_1$ space, then $M^{\perp}$ is a dual $\mathscr{L}_\infty$ space and thus $M^{\perp}$ is complemented in X^*. Moreover, X/M has the B.A.P. The result now follows by 2.4. □

Remarks. (1) Let E be a separable Banach space. By [9], there exists a space Y such that Y^{**} has a basis and Y^{**}/Y is isomorphic to E. If we choose E to be a separable Banach space without the A.P. (see [3]), we have an example of a couple of spaces $Y = M$, $Y^{**} = X$ such that $M^{\perp}$ is complemented in X^*, X and M have the B.A.P. but X/M does not have the A.P. (2) There is apparently no known example of a Banach space X with the A.P. containing a closed subspace M without the A.P., but such that X/M has the A.P. (3) If M is an M-ideal in X, then obviously $M^{\perp}$ is complemented in X^* and thus 2.4 applies. Let us mention that under that assumption if X/M is separable and has the B.A.P., then M is complemented in X (see [14]).

References

1. B. Beauzamy, *Introduction to Banach spaces and their geometry*, North-Holland Math. Studies, 68, Notas Math., 1982.
2. D. Dean, *Approximations and weak star approximation in Banach spaces*, Bull. Amer. Math. Soc. **79** (1973), 725–728.
3. P. Enflo, *A counterexample to the approximation property in Banach spaces*, Acta Math. **130** (1973), 309–317.
4. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. no. 16 (1955).
5. S. Heinrich, *Ultra products in Banach space theory*, J. Reine Angew. Math. **313** (1980), 72–104.
6. W. B. Johnson and H. P. Rosenthal, *On w^*-basic sequences and their applications to the study of Banach spaces*. Studia Math. **43** (1972), 77–92.

7. W. B. Johnson, H. P. Rosenthal and M. Zippin, *On bases, finite-dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506.

8. A. Lima, *Uniqueness of Hahn-Banach extensions and lifting of linear dependences*, Math. Scand. **53** (1983), 97–113.

9. J. Lindenstrauss, *On James' paper "separable conjugate spaces"*, Israel J. Math. **9** (1971), 279–284.

10. J. Lindenstrauss and H. P. Rosenthal, *The $\mathscr{L}_p$ spaces*, Israel J. Math. **7** (1969), 325–349.

11. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*. I: *Sequence spaces*, Springer Verlag, 1977.

12. W. Lusky, *A note on Banach spaces containing c_0 or C_∞*, J. Funct. Anal. **62** (1985), 1–7.

13. A. Szankowski, *$B(H)$ does not have the approximation property*, Acta Math. **147** (1981), 1-2, 89–108.

14. T. Ando, *A theorem on nonempty intersection of convex sets and its application*, J. Approx. Theory **13** (1975), 158–166.

Equipe d'Analyse Université Paris VI, Tour 46-0.4 ° étage, 4 Place Jussieu, 75230 Paris Cedex 05, France

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000 Israel