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POINT DERIVATIONS ON FUNCTION ALGEBRAS GENERATED BY HOLOMORPHIC FUNCTIONS

R. G. M. BRUMMELHUIS AND P. J. DE PAEPE

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ABSTRACT. It is shown that a continuous point derivation on the algebra $H(X)$ consisting of uniform limits on X of functions holomorphic in a neighborhood of a compact subset X in $\mathbf{C}^n$, which vanishes on the polynomials is the trivial derivation.

1. INTRODUCTION

Let X be a compact Hausdorff space. A function algebra B on X is a point separating sup norm closed subalgebra of $C(X)$ (the algebra of all complex-valued continuous functions on X) containing the constant functions on X. ΔB will denote the maximal ideal space of B, i.e. the space of nontrivial complex homomorphisms of B.

A continuous point derivation of B at some element $\phi \in \Delta B$ is a continuous linear functional D on B such that

$$Dfg = \phi(f)Dg + \phi(g)Df$$

for all $f, g \in B$. The collection of all continuous point derivations of B at ϕ is a linear subspace $\mathscr{D}(B, \phi)$ of B^*.

For a compact subset X of $\mathbf{C}^n$, $P(X)$ will denote the uniform closure in $C(X)$ of the polynomials (considered as functions on X) and $H(X)$ will be the closure in $C(X)$ of the collection of functions holomorphic in some neighborhood of X.

Using results and ideas of [BdP] we will prove a result which shows that a continuous point derivation of $H(X)$ is completely determined by its values on the polynomials:

Theorem. *Let X be a compact subset of $\mathbf{C}^n$. Let D be a continuous point derivation of $H(X)$ such that $D \mid P(X) = 0$. Then $D = 0$.*

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2. Proof of the theorem

Let U be an open subset of $\mathbf{C}^n$, containing X. Let $E(U)$ be the envelope of holomorphy of U (see [GR] for basic facts on the theory of several complex variables). Every element f of the algebra $\mathscr{O}(U)$ of holomorphic functions on U can be extended in a unique way to an element $\hat{f}$ of $\mathscr{O}(E(U))$. Now let $\phi \in \Delta H(X)$. The map $f \to \phi(f \mid X)$ from $\mathscr{O}(E(U))$ onto $\mathbf{C}$ defines a continuous homomorphism of $\mathscr{O}(E(U))$ (endowed with the topology of uniform convergence on compact subsets of $E(U)$), and since these are the point evaluations at the points of $E(U)$, [GR, Chapter I, §G], there is a point $a \in E(U)$ such that

$$\phi(f \mid X) = f(a)$$

for all $f \in \mathscr{O}(E(U))$. Now let $D \in \mathscr{D}(H(X), \phi)$ with $D \mid P(X) = 0$. Defining $d: \mathscr{O}(E(U)) \to \mathbf{C}$ by $df = D(f \mid X)$ it follows that d is linear and satisfies

$$d(fg) = f(a)dg + g(a)df$$

for all $f, g \in \mathscr{O}(E(U))$.

Moreover $d(\hat{p}) = 0$ for any polynomial p on $\mathbf{C}^n$. We now argue as in [BdP, Proof of Theorem 1]. By [H, Theorem 5.3.9, p. 128] there exist $f_1, \dots, f_m \in \mathscr{O}(E(U))$ such that the map $\Phi = (\hat{z}_1, \dots, \hat{z}_n, f_1, \dots, f_m): E(U) \to \mathbf{C}^{n+m}$ is one-to-one and proper. Applying [GR, Theorem 15, Chapter VIII, §A, p. 224] to the ideal sheaf of $\{a\}$, we find that for any $f \in \mathscr{O}(E(U))$ there exist functions $h_1, \dots, h_n, g_1, \dots, g_m \in \mathscr{O}(E(U))$ such that

$$f - f(a) = \sum_{i=1}^{n} (\hat{z}_i - \hat{z}_i(a))h_i + \sum_{i=1}^{m} (f_i - f_i(a))g_i. \tag{1}$$

Applying d to (1), using $d\hat{z}_i = 0$, $i = 1, \dots, n$, we obtain

$$df = \sum_{i=1}^{m} g_i(a)df_i, \tag{2}$$

i.e. for any $f \in \mathscr{O}(E(U))$ df is a linear combination of the numbers $df_1, \dots, df_m$. Now let $k \in \mathbf{N}$, $H \in \mathscr{O}(\mathbf{C}^k)$, $b \in \mathbf{C}^k$. Then there are functions $H_1, \dots, H_k \in \mathscr{O}(\mathbf{C}^k)$ such that

$$H(\zeta) = H(b) + \sum_{i=1}^{k} (\zeta_i - b_i)H_i(\zeta), \tag{3}$$

again by the above cited theorem in [GR], or more simply in this case, using power series. Note: $H_i(b) = \frac{\partial H}{\partial \zeta_i}(b)$, $i = 1, \dots, n$. Now in (3) we take $k = n + m$, $\zeta = \Phi(z)$, $b = \Phi(a)$ and $H \in \mathscr{O}(\mathbf{C}^{n+m})$, and apply d.

It follows that with standard coordinates $(z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbf{C}^n \times \mathbf{C}^m = \mathbf{C}^{n+m}$,

$$dH(\hat{z}_1, \dots, \hat{z}_n, f_1, \dots, f_m) = \sum_{i=1}^{m} \frac{\partial H}{\partial w_i}(\Phi(a))\, df_i. \tag{4}$$

Applying Theorem A of Cartan [GR, Theorem 13, Chapter VIII, §A, p. 243] to the sheaf of ideals of $\widetilde{V} = \Phi(E(U))$, it can be shown that given $x \in \widetilde{V}$ there are $H_1, \dots, H_m \in \mathscr{O}(\mathbf{C}^{n+m})$ with H_i vanishing on $\widetilde{V}$ $(i = 1, \dots, m)$ such that $(\partial(H_1, \dots, H_m)/\partial(w_1, \dots, w_m))(x) \neq 0$. In particular for the point $a \in E(U)$ we find $H_1, \dots, H_m \in \mathscr{O}(\mathbf{C}^{n+m})$, vanishing on $\widetilde{V}$ such that

$$\frac{\partial(H_1, \dots, H_m)}{\partial(w_1, \dots, w_m)}(\Phi(a)) \neq 0. \tag{5}$$

By (4) $\sum_{i=1}^m \frac{\partial H_j}{\partial w_i}(\Phi(a))df_i = 0$, $j = 1, \dots, m$.

From (5) it now follows that $df_i = 0$, $i = 1, \dots, m$, so by (2): $df = 0$ for all $f \in \mathscr{O}(E(U)$. So $D|\mathscr{O}(U)|X = 0$, so by continuity $D = 0$.

Remark. For the special case that X is holomorphically convex, i.e. $\Delta H(X) = X$, we can give a short function algebraic proof of the theorem: First, we recall a well-known fact: if B is a function algebra on X with $P(X) \subset B \subset H(X)$ and with $\Delta B = X$, then $B = H(X)$. Indeed: the functions $z_1, \dots, z_n$ belong to B and the joint spectrum $\sigma_B(z_1, \dots, z_n)$ equals X since $\Delta B = X$. So if $f \in \mathscr{O}(U)$, U open in $\mathbf{C}^n$ and containing X, by the functional calculus [G2] $f \circ (\hat{z}_1, \dots, \hat{z}_n) \in \widehat{B}$ (here $\hat{g}$ denotes the Gelfand transform of $g \in B$), so $f \in B$. Hence $\mathscr{O}(U)|X \subset B$, so $B = H(X)$. Now assume $\Delta H(X) = X$ and D is a continuous point derivation on $H(X)$ with $D|P(X) = 0$. Let $B = \operatorname{Ker} D$. The kernel of D is a function algebra on X containing $P(X)$ and $\Delta D = \Delta H(X)$, cf. [G1]. By the above remark: $\operatorname{Ker} D = H(X)$, i.e. $D = 0$.

3. Examples

Since the theorem shows that any $D \in \mathscr{D}(H(X), \phi)$ is completely determined by its values on $P(X)$, it follows that D is completely determined by $D(z_1), \dots, D(z_n)$. This shows

Corollary. *Let X be a compact subset of $\mathbf{C}^n$ and $\phi \in \Delta H(X)$. Then*

$$\dim \mathscr{D}(H(X), \phi) \leq n.$$

We give some simple examples.

(i) If X is a compact subset of $\mathbf{C}^n$ and a belongs to the interior of X, then for any $c_1, \dots, c_n \in \mathbf{C}$, $\sum_{i=1}^n c_i \frac{\partial}{\partial z_i}\big|_a$ belongs to $\mathscr{D}(H(X), a)$, i.e. for an interior point $a \in X$, $\dim \mathscr{D}(H(X), a) = n$.

(ii) Let $X = \{z \in \mathbf{C}: |z| \leq 1\}$, then $P(X) = H(X)$. The continuous point derivations D at points a with $|a| < 1$ are by example (i) of the form $Df = cf'(a)$, $c \in \mathbf{C}$. Let a be a boundary point of X, say $a = 1$. Consider the function $f = \sqrt{1 - z} \in H(X)$. If $D \in \operatorname{Der}(H(X), 1)$, $Dz = -D(1 - z) = -Df^2 = 2f(1)Df = 0$. Hence $D = 0$.

(iii) In a similar way, it follows that for

$X = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq 1, |z_1| \leq 1\}$,

$\dim \mathscr{D}(H(X), (a_1, a_2)) = 2$ if $|a_1| < 1$, $|a_2| < 1$,

$\dim \mathscr{D}(H(X), (a_1, a_2)) = 1$ if $|a_1| = 1$, $|a_2| < 1$ or $|a_2| = 1$, $|a_1| < 1$,

$\dim \mathscr{D}(H(X), (a_1, a_2)) = 0$ if $|a_1| = |a_2| = 1$.

(iv) Let $X = \{(z_1, z_2) \in \mathbf{C}^2 : z_1^3 = z_2^2, |z_1| \leq 1\}$. Suppose D is a continuous point derivation of $P(X) = H(X)$ at $(a_1, a_2) \in X$. Since $z_1^3 = z_2^2$ on X we have $3a_1^2 Dz_1 = 2a_2 Dz_2$ so if $(a_1, a_2) \neq (0, 0)$, $Dz_2 = (3a_1^2/2a_2)Dz_1$, so it follows that

$$\dim \mathscr{D}(H(X), (a_1, a_2)) = 1 \qquad \text{if } (a_1, a_2) \in X, 0 < |a_1| < 1$$

and

$$\dim \mathscr{D}(H(X), (a_1, a_2)) = 0 \qquad \text{if } (a_1, a_2) \in X, |a_1| = 1.$$

Finally, $\mathscr{D}(H(X), (0, 0))$ has dimension 2: if $f \in H(X)$, $g(z) = f(z^2, z^3)$, $|z| \leq 1$, then $D_1 f = g''(0)$, $D_2 f = g'''(0)$ define two linearly independent elements of $\mathscr{D}(H(X), (0, 0))$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AMSTERDAM, ROETERSSTRAAT 15, 1018 WB AMSTERDAM, THE NETHERLANDS