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INFINITE VERTEX-TRANSITIVE, EDGE-TRANSITIVE NON-1-TRANSITIVE GRAPHS

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ABSTRACT. We show that every vertex-transitive, edge-transitive graph of odd valence and subexponential growth is 1-transitive, thus extending to infinite graphs a theorem of W. T. Tutte for finite graphs. We describe a number of counterexamples in the case of exponential growth.

1. INTRODUCTION

In this note the symbol Γ will denote an undirected simple graph; Γ may be finite or infinite. In [5, p. 59, item 7.53], W. T. Tutte proved that every finite, vertex-transitive, edge-transitive graph Γ of odd valence is 1-transitive. He inquired whether his result still held when the valence of Γ is even. A negative answer to Tutte's query was provided by I. Z. Bouwer with an infinite family of finite counterexamples [1].

The main result of the present note is that Tutte's theorem holds for infinite graphs, provided that their growth is *subexponential*, that is, if $d(k)$ denotes the number of vertices at distance k from some fixed vertex and if a is any real number such that $a > 1$, then $\liminf(d(k)/a^k) = 0$. Several families of examples in the final section show that if the growth is exponential, then Tutte's theorem is not necessarily extendable, regardless of the parity of the valence. Another example illustrates the indispensability of the condition of odd valence, even when the growth is subexponential.

2. PRELIMINARIES

The symbols $V(\Gamma)$, $E(\Gamma)$, and $\rho(\Gamma)$ (or just ρ) will denote, respectively, the vertex set, the edge set, and the valence (when constant on $V(\Gamma)$) of Γ. We say that Γ is 1-*transitive* if given $\{x_i, y_i\} \in E(\Gamma)$ $(i = 1, 2)$, there exists an automorphism φ of Γ such that $\varphi(x_1) = x_2$ and $\varphi(y_1) = y_2$. If $S \subseteq V(\Gamma)$, then $\partial(S)$ denotes the set of vertices in $V(\Gamma)\backslash S$ which are adjacent to at least

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one vertex of S. When S is a set of vertices in a directed graph, we define $\partial(S)$ similarly, with no regard for the orientation of the edges.

We use the notion of an "end" in a graph as formulated by R. Halin [2]. In the special case of an infinite, locally finite graph Γ, the number of ends $\varepsilon(\Gamma)$ of Γ may be defined as the supremum of the number of infinite components of $\Gamma - S$, where S ranges over all finite subsets of $V(\Gamma)$. By combining [3, Corollary 15] and [4, Theorem 1], one has that $\varepsilon(\Gamma)$ equals 1 or 2 or is uncountable whenever Γ is infinite, locally finite, connected, and vertex-transitive.

3. Vertex-transitive, edge-transitive graphs having subexponential growth

Lemma 1. *Let Δ be a directed graph with constant in-valence ρ^- and constant out-valence ρ^+, and suppose that $1 \le \rho^- < \rho^+$. For any finite subset S of $V(\Delta)$, one has*

$$|\partial S| \ge 2|S|/(\rho - 1),$$

where $\rho = \rho^+ + \rho^-$.

Proof. The number of edges with initial vertex in S is $\rho^+|S|$. Among these edges, at most $\rho^-|S|$ also have their terminal vertex in S. Hence the number of edges from S to ∂S is at least $(\rho^+ - \rho^-)|S| \ge |S|$. Since at most ρ^- of these edges share any common terminal vertex in ∂S, we have

$$|\partial S| \ge (1/\rho^-)|S| \ge (2/(\rho - 1))|S|. \quad \square$$

Lemma 2. *Assume the hypothesis of Lemma 1, and moreover, that Δ has no 2-cycle and $\rho^+ + \rho^- = \rho \ge 3$. Let $x \in V(\Delta)$, and let V_k denote the set of vertices whose undirected distance from x is k for $k = 0, 1, \ldots$. Then*

$$|V_k| \ge 2((\rho + 1)/(\rho - 1))^{k-1}$$

for all $k \ge 1$.

Proof. Noting that $|V_0| = 1$ and $|V_1| = \rho$, we proceed by induction by letting m be a positive integer and supposing the above inequality to hold whenever $1 \le k \le m$. Clearly $V_{m+1} = \partial(V_0 \cup V_1 \cup \cdots \cup V_m)$. Hence by Lemma 1,

$$\begin{aligned} |V_{m+1}| &\ge \frac{2}{\rho - 1} \sum_{k=0}^{m} |V_k| \\ &\ge \frac{2}{\rho - 1} \left[1 + \rho + 2 \sum_{k=2}^{m} \left(\frac{\rho + 1}{\rho - 1} \right)^{k-1} \right] \\ &= 2 \left(\frac{\rho + 1}{\rho - 1} \right)^{m}. \quad \square \end{aligned}$$

Theorem. *Let Γ be a connected infinite, vertex-transitive, edge-transitive graph of odd valence. Let $d(k)$ be the number of vertices of Γ at distance k from any given vertex of Γ. If the function $d(k)$ is subexponential, then Γ is 1-transitive.*

Proof. (Our proof is parallel to Tutte's proof in the finite case up to the point where Tutte uses the equality $\rho|V|/2 = |E|$.) Let (x, y) be an ordered pair of

adjacent vertices of Γ and let $M = \{(\varphi(x), \varphi(y)) : \varphi$ is an automorphism of $\Gamma\}$. Since Γ is edge-transitive, it follows that if $\{u, v\} \in E(\Gamma)$, then $(u, v) \in M$ or $(v, u) \in M$. Moreover, Γ is 1-transitive if and only if there exists an edge $\{u, v\} \in E(\Gamma)$ such that both $(u, v) \in M$ and $(v, u) \in M$. Suppose Γ is not 1-transitive. Thus for each $\{u, v\} \in E(\Gamma)$, exactly one of (u, v) and (v, u) belongs to M. Consider the directed graph Δ with $V(\Delta) = V(\Gamma)$ and $E(\Delta) = M$, obtained by orienting the edges of Γ. Thus Δ is both vertex- and edge-transitive and has constant in-valence ρ^- and constant out-valence ρ^+. Since $\rho(\Gamma)$ is odd, $\rho^+ \neq \rho^-$. We lose no generality by assuming $\rho^- < \rho^+$; otherwise replace (x, y) by (y, x) at the outset.

By Lemma 2, the function $d(k)$ for Δ grows exponentially. But Γ has the very same growth function, contrary to hypothesis. Hence Γ is 1-transitive. □

4. The case of exponential growth

Bouwer [1] described an infinite class of finite connected vertex-transitive, edge-transitive, non-1-transitive graphs. If B denotes one of these graphs and Γ is the connected graph such that every block of Γ is isomorphic to B and such that each vertex of Γ is in p blocks, then Γ is vertex-transitive, edge-transitive, non-1-transitive and $\rho(\Gamma) = p\rho(B)$, which is even. A simpler example is obtained by replacing the above B by $K_{m,n}$ where $2 \leq m < n$. Furthermore we write $p = p_1 + p_2$ where $1 \leq p_1 \leq p_2$. Now, if every vertex of Γ is in p blocks such that it is on the "n-side" of p_1 copies $K_{m,n}$ and on the "m-side" of p_2 copies, then Γ is vertex-transitive, edge-transitive and non-1-transitive and $\rho(\Gamma) = p_1 m + p_2 n$ which may be even or odd. The lexicographic product of Γ and the empty graph on k vertices is a k-connected graph with the same properties. All the above graphs have infinitely many ends. Examples with only one end and with odd valence may be obtained by taking one of the above examples Γ of odd valence and forming the cartesian product $\Gamma \times \Gamma \times \Gamma$ (for which $\rho(\Gamma \times \Gamma \times \Gamma) = 3\rho(\Gamma)$).

In order to complete the discussion we note that our theorem does not hold for infinite connected graphs of even valence and subexponential growth, in fact not even for graphs of constant growth. This can be seen by a modification of one of Bouwer's examples. We let $V(\Gamma_0) = \mathbf{Z} \times \mathbf{Z}_9$, where $\mathbf{Z}$ denotes the ring of integers and $\mathbf{Z}_9$ is the ring of integers modulo 9. For $i \in \mathbf{Z}$, let $r(i)$ denote the residue of i modulo 6. The set $E(\Gamma_0)$ will consist of all 2-subsets of $V(\Gamma_0)$ of the following two types: $\{(i, a), (i+1, a)\}$ and $\{(i, a), (i+1, a+2^{r(i)})\}$, where $i \in \mathbf{Z}$, $a \in \mathbf{Z}_9$ and arithmetic is executed in the appropriate ring. (Thus Bouwer's smallest counterexample $X(2, 6, 9)$ on 54 vertices is a quotient graph of Γ_0 obtained by identifying vertices whose first coordinates are congruent modulo 6.) Note that $\rho(\Gamma_0) = 4$; each vertex is incident with two edges of each type. The proof that Γ_0 is vertex- and edge-transitive is identical to [1, Proposition 1]. To see that Γ_0 is not 1-transitive we first observe that the only vertex sets S of Γ_0 with at most 9 vertices and the property that $\Gamma_0 - S$

has two infinite components are the sets $S_i = \{(i, a) | a \in \mathbf{Z}_9\}$. So if Γ_0 has an automorphism φ such that $\varphi((1, 0)) = (2, 0)$ and $\varphi((2, 0)) = (1, 0)$, then $\varphi(S_1) = S_2$, $\varphi(S_2) = S_1$ and $\varphi(S_3) = S_0$. Then the 6-cycle with vertices $(1, 0)$, $(2, 0)$, $(3, 4)$, $(2, 4)$, $(1, 2)$, $(2, 2)$ would be mapped into a 6-cycle with one vertex in S_0, two vertices in S_2, and three vertices in S_1. But it is easy to see that Γ_0 has no such 6-cycle, and hence Γ_0 is not 1-transitive.

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