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COEFFICIENTS OF SYMMETRIC FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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ABSTRACT. The well-known inclusion relation between functions with bounded boundary rotation and close-to-convex functions of some order is extended to m-fold symmetric functions. This leads solving the corresponding result for close-to-convex functions to the sharp coefficient bounds for m-fold symmetric functions of bounded boundary rotation at most $k\pi$ when $k \geq 2m$. Moreover it shows that an m-fold symmetric function of bounded boundary rotation at most $(2m+2)\pi$ is close-to-convex and thus univalent.

1. INTRODUCTION

We consider functions which are analytic in the unit disk $\mathbf{D}$. By P we denote the family of functions p which have the normalization

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1}$$

and have positive real part; by $\widetilde{P}$ we denote the family of functions p which are normalized by (1) and there exists a complex number a such that the rotated function ap has positive real part.

We consider functions f which have the usual normalization

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

A function is called m-fold symmetric if it has the special form $(m \in \mathbf{N})$,

$$f(z) = z + a_{m+1} z^{m+1} + a_{2m+1} z^{2m+1} + \cdots . \tag{2}$$

By K_m, St_m, $C_m(\beta)$ and $V_m(k)$ respectively we denote the families of m-fold symmetric convex, starlike, close-to-convex functions of order β and functions of bounded boundary rotation at most $k\pi$, respectively. A function is called convex or starlike if it maps the unit disk univalently onto a convex or starlike domain respectively.

A function f is called close-to-convex of order β, $\beta \geq 0$, if there is a convex function φ such that $f'/\varphi' = p^\beta$ for some function $p \in \widetilde{P}$. For $\beta \leq 1$ it turns out that a function is close-to-convex or order β, if and only if it maps

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D univalently onto a domain whose complement E is the union of rays, which are pairwise disjoint up to their tips, such that every ray is the bisector of a sector of angle $(1-\beta)\pi$ which wholly lies in E (see e.g. [2], and [12, p. 176]). By means of the introductory paper of Kaplan [7], it is easily verified that for an m-fold symmetric function f the corresponding function φ can be chosen also to be m-fold symmetric. This observation is due to Pommerenke [11], who studied coefficient problems in $C_m(\beta)$. His asyptotic results give support to the conjecture that if $\beta > 1 - 2/m$, then the coefficients of a function $f \in C_m(\beta)$ given by (2) are dominated in modulus by the corresponding coefficients of the function g given by

$$(3) \qquad g'(z) = \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}}, \qquad g(0) = 0 .$$

Coefficient domination is denoted by $f \ll g$.

The above statement had been settled for $m = 1$ by Brannan, Clunie and Kirwan [4] and the final step by Aharnov and Friedland [1] and independently by Brannan [3], (see e.g. [14, Chapter 2]), and for $\beta = 1$ by Pommerenke [11, Theorem 3]. This latter statement includes the truth of the Littlewood-Paley conjecture (see e.g. [6, §3.8]) for odd close-to-convex functions (of order one).

In §2 we give a proof of the above statement for $\beta \geq 1 - 1/m$, whereas for $0 < \beta < 1 - 1/m$ the statement is false as examples show, so that the number $1 - 1/m$ is sharp. However, for $\beta = 0$, i.e. for convex functions, the statement is again true, as was shown by Robertson [13, p. 380].

The boundary rotation of a function f is defined by

$$\sup_{0<r<1} \int_0^{2\pi} \left| \operatorname{Re} \left(1 + \frac{zf''}{f'} \right) (re^{i\theta}) \right| d\theta .$$

Paatero [10] showed that $f \in V_1(k)$, if and only if

$$1 + \frac{zf''}{f'} = \left(\frac{k}{4} + \frac{1}{2} \right) \cdot p_1 - \left(\frac{k}{4} - \frac{1}{2} \right) \cdot p_2$$

for some $p_1, p_2 \in P$. An inspection of Paatero's proof shows that for an m-fold symmetric function, p_1 and p_2 can be chosen to have the form

$$(4) \qquad p_{1,2}(z) = 1 + c_m z^m + c_{2m} z^{2m} + \cdots .$$

It is well known [4], (see e.g. [17, Theorem 2.26]) that functions of bounded boundary rotation are close-to-convex of some order, namely

$$V_1(k) \subset C_1(k/2 - 1) .$$

In §3 we give an improvement of this result for m-fold symmetric functions:

$$V_m(k) \subset C_m((k/2 - 1)/m) ,$$

which leads to the solution of the coefficient problem for m-fold symmetric functions of bounded boundary rotation when $k \geq 2m$. This result includes the truth of the Littlewood-Paley conjecture for odd functions of bounded boundary rotation 6π.

2. The coefficients of symmetric close-to-convex functions.

Here we shall prove

Theorem 1. *Let* $m \in \mathbf{N}$, $\beta \geq 1 - 1/m$ *and* $f \in C_m(\beta)$. *Then*

$$f' \ll \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}}.$$

Proof. Let f be an m-fold symmetric close-to-convex function of order β. Then there exist $\varphi \in K_m$ and $p \in \widetilde{P}$ such that

$$f'(z) = \varphi'(z) \cdot p^\beta(z^m).$$

For each $\varphi \in K_m$ there is a $g \in St_m$ such that $g = z\varphi'$ (see e.g. [14, Theorem 2.4]), for which there is a representation of the form (see [5, Theorem 3])

$$g(z) = \int_{|x|=1} \frac{z}{(1-xz^m)^{2/m}}\, d\mu,$$

where μ is a Borel probability measure on the unit circle. Thus we have

$$\begin{aligned} f'(z) &= \int_{|x|=1} \frac{d\mu}{(1-xz^m)^{2/m}} \cdot p^\beta(z^m) \\ &= \int_{|x|=1} \frac{d\mu}{(1-x^2z^{2m})^{1/m}} \cdot \left(\frac{1+xz^m}{1-xz^m}\right)^{1/m} \cdot p^\beta(z^m). \end{aligned}$$

For fixed $x \in \partial\mathbf{D}$ the function

$$\left(\left(\frac{1+xz^m}{1-xz^m}\right)^{1/m} \cdot p^\beta(z^m)\right)^{1/(\beta+1/m)} =: q_x(z^m)$$

is of the form (4) and lies in $\widetilde{P}$. A well-known lemma [4, 3], (see e.g. [14, Theorem 2.21]) implies that

$$q_x^{\beta+1/m}(z^m) \ll \left(\frac{1+z^m}{1-z^m}\right)^{\beta+1/m},$$

because $\beta + 1/m \geq 1$. Thus we get

$$\begin{aligned} f'(z) &= \int_{|x|=1} \frac{d\mu}{(1-x^2z^{2m})^{1/m}} \cdot q_x^{\beta+1/m}(z^m) \\ &= \sum_{j=0}^{\infty} \binom{j-1+1/m}{j} z^{2mj} \left\{\int_{|x|=1} x^{2j} q_x^{\beta+1/m}(z^m)\, d\mu\right\} \\ &\ll \sum_{j=0}^{\infty} \binom{j-1+1/m}{j} z^{2mj} \left(\frac{1+z^m}{1-z^m}\right)^{\beta+1/m} = \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}}, \end{aligned}$$

because μ has total mass one and all numbers $\binom{j-1+1/m}{j}$ are nonnegavite. □

We remark that the result is sharp, because the function g defined by (3) is in $C_m(\beta)$ (see e.g. [11, p. 264]).

For $0 < \beta < 1 - 2/m$ Pommerenke showed [11, Theorem 2], that $a_n = o(1/n)$ for a function $f \in C_m(\beta)$, and that this cannot be improved [11, p. 265]. But on the other hand, for $\beta > 1 - 2/m$,

$$a_n = O(n^{\beta-2+2/m}),$$

[11, Theorem 1].

Neverthelelss, the statement of Theorem 1 is not true in the case $1 - 2/m < \beta < 1 - 1/m$, not even for the third nonvanishing coefficient a_{2m+1}, as the following examples show. For $0 \le t \le 1$ let

$$f'(z) = \frac{1}{(1-z^m)^{2/m}} \cdot \left(t\left(\frac{1+z^m}{1-z^m}\right) + (1-t)\left(\frac{1+z^{2m}}{1-z^{2m}}\right)\right).$$

Then obviously $f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots \in C_m(\beta)$. It follows that

$$(2m+1)a_{2m+1} = 2\beta(1 + (\beta-1)t^2) + \frac{4\beta t}{m} + \frac{1}{m}\left(1 + \frac{2}{m}\right) =: F(t).$$

The relation $F'(t_0) = 0$ implies that

$$t_0 = \frac{1}{m(1-\beta)},$$

which lies between 0 and 1 if $0 < \beta < 1 - 1/m$, so that F has a local maximum at t_0, which is greater than the corresponding coefficient of g, as is easily seen.

3. The coefficients of symmetric functions of bounded boundary rotation.

It is well known that functions of bounded boundary rotation are close-to-convex of some order,

$$V_1(k) \subset C_1(k/2 - 1). \tag{5}$$

We shall give now a generalized version of this statement for m-fold symmetric functions. We need the following

Lemma. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and $h(z) = z + b_{m+1}z^{m+1} + b_{2m+1}z^{2m+1} + \cdots$ have the property*

$$h'(z) = (f'(z^m))^{1/m}.$$

Then

$$f \in V_1(k) \Leftrightarrow h \in V_m(k)$$

and

$$f \in C_1(\beta) \Leftrightarrow h \in C_m(\beta/m).$$

Proof. Let $f \in V_1(k)$. Then

$$1 + \frac{z^m f''(z^m)}{f'(z^m)} = 1 + \frac{zh''(z)}{h'(z)},$$

so that $h \in V_m(k)$, and conversely.

If $f \in C_1(\beta)$, then there are $\varphi \in K_1$ and $p \in \widetilde{P}$ such that

$$f'(z) = \varphi'(z) \cdot p^\beta(z).$$

Now

$$\begin{aligned} h'(z) = (f'(z^m))^{1/m} &= (\varphi'(z^m))^{1/m} \cdot p^{\beta/m}(z^m) \\ &= \varphi_1(z) \cdot p^{\beta/m}(z^m). \end{aligned}$$

The function φ_1 represents an m-fold symmetric convex function, because a function is convex, if and only if $1 + zf''/f' \in P$ (see e.g. [14, Theorem 2.4]), and

$$1 + \frac{z\varphi_1''(z)}{\varphi_1'(z)} = 1 + \frac{z^m \varphi''(z^m)}{\varphi'(z^m)}.$$

So it follows that $h \in C_m(\beta/m)$, and conversely. □

We remark that the lemma can be used to show that Theorem 1 with $\beta = 1/2$, $m = 2$ is somewhat stronger than the case $\beta = 1$, $m = 1$. For example it leads to the estimates $||a_3| - |a_2|| \le 1$ and $||a_4| - |a_2|| \le 2$ for close-to-convex functions [8, 9].

An application of the lemma, with the aid of (5), gives

Theorem 2. *Let* $m \in \mathbf{N}$, $k \ge 2$. *Then*

$$V_m(k) \subset C_m((k/2 - 1)/m).$$

This leads to the following statements

Theorem 3. *Let* $m \in \mathbf{N}$, $k \ge 2m$ *and* $f \in V_m(k)$. *Then*

$$f' \ll \frac{(1 + z^m)^{(k/2-1)/m}}{(1 - z^m)^{(k/2+1)/m}}.$$

This follows with Theorem 1. Observe that the statement is sharp, because the functions defined by (3) with $\beta = (k/2 - 1)/m$ are in $V_m(k)$,

$$1 + \frac{zg''}{g'}(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \cdot \frac{1 + z^m}{1 - z^m} - \left(\frac{k}{4} - \frac{1}{2}\right) \cdot \frac{1 - z^m}{1 + z^m}.$$

For $m = 2$, $k = 6$ we have the statement of the Littlewood-Paley conjecture. Another example is $m = 2$, $k = 4$. Here one gets the sharp bounds for f, normalized by (2),

$$|a_{2n+1}| \le \begin{cases} \dfrac{1}{2n+1}\left(\dbinom{n/2 + 1/2}{n/2} + \dbinom{n/2 - 1/2}{n/2 - 1}\right) & \text{if } n \text{ is even}, \\[2ex] \dfrac{2}{2n+1}\dbinom{n/2}{n/2 - 1/2} & \text{if } n \text{ is odd}. \end{cases}$$

It is an open question if the statement of Thoerem 3 remains true, when $k < 2m$. The close-to-convex counterexamples, given after Theorem 1, cannot be used here.

Furthermore we have

Theorem 4. *Let* $m \in \mathbf{N}$. *Then* $V_m(2m+2)$ *consists of close-to-convex and thus univalent functions.*

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