PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

FUNCTIONS OF EXPONENTIAL TYPE AND SEPARATED SEQUENCES

DAVID C. ULLRICH

(Communicated by J. Marshall Ash)

Let us say that the sequence of complex numbers (λ_j) is *separated* if there exists $\varepsilon > 0$ such that $|\lambda_j - \lambda_k| \geq \varepsilon \ (j \neq k)$. For $0 < p < \infty$ let E^p denote the class of entire functions of exponential type whose restriction to the real line lies in the Lebesgue space L^p. A subharmonicity argument establishes the following classical result:

Theorem A. *Suppose (λ_j) is a separated sequence of complex numbers with bounded imaginary parts. Then*

$$\sum |f(\lambda_j)|^p < \infty \tag{1}$$

for every $f \in E^p$.

(Theorem 17 on p. 96 of [Y] asserts this under the additional hypothesis that the λ_j be real, but the argument in [Y] proves the slightly more general statement above.)

The purpose of this note is to point out that this result is sharp in a certain sense, settling a question posed on p. 221 of [Y]:

Theorem 1. *Suppose $p > 0$ and (λ_j) is a sequence of complex numbers satisfying (1) for each $f \in E^p$. Then the λ_j have bounded imaginary parts and can be partitioned into a finite union of separated sequences.*

The proof will be a simple example of what is sometimes referred to as a "gliding hump" argument:

Proof. Suppose the λ_j satisfy (1) for every $f \in E^p$. We leave it to the reader to show that $\mathrm{Im}(\lambda_j)$ is bounded; this is analogous to the argument below, but simpler. Supposing this, we show that the λ_j can be partitioned into a finite union of separated sequences. We need only show that the number of λ_j with real part lying in the interval $[\beta, \beta + 1]$ is bounded. Supposing that this is false, we shall construct an $f \in E^p$ for which (1) fails: Choose $A \geq 1$ so that $|\mathrm{Im}(\lambda_j)| \leq A$ for all j. Now choose $g \in E^p$ with $|g(z)| \geq 1$ for all $z \in \mathbf{C}$

Received by the editors July 6, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42A70; Secondary 30D55.

with $|z| \le 2A$. We claim that if constants $c_k > 0$ and $\beta_k \in \mathbf{R}$ are chosen properly, then $f(z) = \sum_k c_k g(z + \beta_k)$ will give the required example.

Note first that if we take c_k decreasing to zero fast enough then we will have $f \in E^p$ regardless of our choice of (β_k). Fix such a sequence (c_k). Let $I_\beta = \{j\colon \operatorname{Re}(\lambda_j) \in [\beta, \beta + 1]\}$, so that our hypothesis on the λ_j becomes the statement that the cardinality of I_β is unbounded. Suppose we have chosen $\beta_1, \dots, \beta_N$ in such a way that

$$(2_N) \qquad \sum_{j \in I_{\beta_k}} |f_N(\lambda_j)|^p > 1 \qquad (k = 1, \dots, N),$$

where we have written $f_N(z) = \sum_{k=1}^N c_k g(z + \beta_k)$. Now, Theorem A shows that $g(z)$ approaches zero as z tends to infinity within the strip $|\operatorname{Im}(z)| \le A$. Thus, merely to take β_{N+1} large enough will accomplish two things: It will ensure that (2_N) remains true, at least for $1 \le k \le N$, when f_N is replaced by f_{N+1}, and it will ensure that $|f_N(\lambda_j)| \le \frac{1}{2}$ for $j \in I_{\beta_{N+1}}$. If in addition to taking β_{N+1} large we choose β_{N+1} so that the cardinality of $I_{\beta_{N+1}}$ is sufficiently large (depending on c_{N+1} and p) then we have

$$(2_{N+1}) \qquad \sum_{j \in I_{\beta_k}} |f_{N+1}(\lambda_j)|^p > 1 \qquad (k = 1, \dots, N+1).$$

Now fix k and let N tend to infinity: $\sum_{j \in I_{\beta_k}} |f(\lambda_j)|^p \ge 1$, so that

$$\sum_j |f(\lambda_j)|^p = \infty.$$

References

[Y] R. M. Young, *An introduction to nonharmonic Fourier series*, Academic Press, New York, 1980.

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078