PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 3, March 1989

REMARK ON WITTEN'S MODULAR FORMS

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(Communicated by J. M. Rosenberg)

ABSTRACT. We give a simple proof of the modular invariance of a power series which Witten [4] attaches to an even-dimensional closed manifold whose first Pontryagin class is torsion. The proof uses only the functional equation satisfied by classical theta functions.

In this note, we give a simple proof of the modular invariance of a modular form which E. Witten attaches to any even-dimensional compact spin manifold [4 and 2]. This modular form was also introduced by Schellekens and Warner [3]. We actually obtain more modular forms, one for any even-dimensional homology class, the weight depending on the degree of the class (see theorem below). All these are classical modular forms, with respect to $\mathrm{SL}(2,\mathbb{Z})$. The first mathematical proof of the modular character of Witten's function is due to Schellekens and Warner [3], and to Zagier [6].

Let then M be a smooth manifold of even dimension $d = 2k$. Let $\widehat{A}(M)$ be the total $\widehat{A}$ genus of M. For E a complex vector bundle on M, and t an indeterminate, let

$$S_t(E) = 1 + t \cdot E + t^2 \cdot S^2(E) + \cdots$$

be a formal power series in t with coefficients in the ring $K(M)$. Notice $S_t(E) = (1 - t \cdot E)^{-1}$ for E a line bundle. Let T be the complexified tangent bundle of M. The modular form of Witten is [4]:

$$F(q) = q^{-d/24} \left\langle \widehat{A}(M) \cdot \mathrm{ch}\left(\bigotimes_{l=1}^{\infty} S_{q^l} T \right), [M] \right\rangle \cdot \eta(q)^d.$$

Zagier [6] indeed proves this is the q-expansion of a (meromorphic) modular form of weight k with respect to $\mathrm{SL}(2,\mathbb{Z})$, under the following assumption:

(A) The First Pontryagin class of M is torsion.

A beautiful physical proof of this, based on Feynman path integration, is given by Witten [5].

Received by the editors March 29, 1988 and, in revised form, May 6, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55N20; Secondary 10D12, 57R75.
Key words and phrases. Spin manifolds, modular forms, theta function.
This work was supported in part by the NSF.

0002-9939/89 $1.00 + $.25 per page

In this note, we prove more generally (this is also implicit in [6]).

Theorem. *Under assumption* (A), *for any homology class* γ *in* $H_{2m}(M)$, *the power series in* q:

$$H(q) = q^{-d/24}\left\langle \widehat{A}(M)\cdot \mathrm{ch}\left(\bigotimes_{l=1}^{\infty} S_{q^l}T\right), \gamma\right\rangle \cdot \eta^d$$

is the q*-expansion of a modular form of weight* m, *with respect to* $\mathrm{SL}(2, \mathbb{Z})$ (*here* $\eta = q^{-1/24}\cdot\prod_{l=1}^{\infty}(1-q^l)$ *is the eta-function*).

For the proof, we express the power series in q, with coefficients in $H^*(M)$:

$$K(q) = q^{-d/24}\widehat{A}(M)\cdot \mathrm{ch}\left(\bigotimes_{l=1}^{\infty} S_{q^l}T\right)\cdot \eta^d$$

in terms of Jacobi's theta function θ. This is a holomorphic function of two complex variables z and τ, where z is arbitrary and τ belongs to the upper half-plane:

$$\theta(z, \tau) = c\cdot q^{1/8}\cdot(2\sin \pi z)\prod_{l=1}^{\infty}(1-q^l e^{2\pi i z})(1-q^l\cdot e^{-2\pi i z}),$$

$$\text{where } q = e^{2\pi i \tau}, \quad c = \prod_{l=1}^{\infty}(1-q^l).$$

The only fact we will need is the well-known transformation formula (see e.g. [1, p. 75]):

$$\sqrt{\frac{\tau}{i}}\cdot\theta(z, \tau) = i\cdot e^{-\pi i z^2/\tau}\cdot\theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right).$$

Now introduce formal cohomology classes $\beta_1, \beta_2, \dots, \beta_k$, such that the Chern classes of T are the elementary symmetric functions of $2\pi i\beta_1$, $-2\pi i\beta_1, 2\pi i\beta_2, -2\pi i\beta_2, \dots$.

We obtain for $K(q)$ the following expression:

$$K(q) = q^{-d/24}\cdot\eta(\tau)^d\cdot\frac{\beta_1\cdots\beta_k\cdot(2\pi i)^k}{2^k\cdot\sinh(\pi i\beta_1)\cdots\sinh(\pi i\beta_k)}$$

$$\cdot\prod_l\prod_{j=1}^{k}(1-q^l e^{2\pi i\beta_j})^{-1}\cdot(1-q^l e^{-2\pi i\beta_j})^{-1}.$$

In terms of θ-functions, this becomes:

$$K(q) = \pi^k\cdot\eta(\tau)^{3d/2}(\beta_1\cdots\beta_k)\cdot\prod_{j=1}^{k}\theta(\beta_j, \tau)^{-1}.$$

The assumption (A) means that $\sum_j \beta_j^2 = 0$ in $H^*(M)$. This implies the simple functional equation:

$$\prod_{j=1}^{k}\theta\left(\frac{\beta_j}{\tau}, \frac{-1}{\tau}\right)^{-1} = \left(\frac{\tau}{i}\right)^{-k/2}\cdot i^k\cdot\prod_{j=1}^{k}\theta(\beta_j, \tau)^{-1}.$$

For $\underline{a} = (a_1, a_2, \dots, a_k)$ a multi-index, let $Q_{\underline{a}}(\tau)$ be the coefficient of $\beta^{\underline{a}} = \beta_1^{a_1} \cdots \beta_k^{a_k}$ in the Taylor expansion of $\beta_1 \cdots \beta_k \cdot \prod_{j=1}^k \theta(\beta_j, \tau)^{-1}$. From the functional equation, we get:

$$Q_{\underline{a}}\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{-3k/2+|\underline{a}|} \cdot i^{|\underline{a}|} \cdot Q_{\underline{a}}(\tau).$$

Recall that $\eta(\tau)$ satisfies: $\eta(-1/\tau) = (\tau/i)^{1/2} \cdot \eta(\tau)$. Since the homogeneous component $K_m(q)$ of degree $2m$ of $K(q)$ is the product of $\pi^k \cdot \eta(\tau)^{3d/2}$ with $\sum_{|\underline{a}|=m} Q_{\underline{a}}(\tau)\beta^{\underline{a}}$, it satisfies:

$$K_m\left(-\frac{1}{\tau}\right) = \tau^m \cdot K_m(\tau).$$

Hence, K_m is indeed a modular form of weight m. Q.E.D.

Remark. (1) It is clear from the proof of the Theorem that if the first Pontryagin class of M is non-torsion, the function $F(q)$ of Witten does not stand much chance of being modular, unless some miraculous cancelling of Pontryagin numbers takes place. There does not seem to be any point in making this more precise here (see [3, and 6]).

(2) From the point of view of the algebraic geometer, θ is a section of a holomorphic line bundle on the "universal" family of elliptic curves. One might therefore hope for an approach to Witten's modular forms based on algebraic geometry, although I have no idea how to do it.

(3) The original elliptic genus of Ochanine, in the form of λ [2], as well as the other modular forms of Witten, may be treated using the four θ-functions of Jacobi.

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