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HYPO-ANALYTIC PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Let Ω be a hypo-analytic manifold of dimension m equipped with a hypo-analytic structure whose structure bundle T' has dimension m. This paper introduces hypo-analytic pseudodifferential operators and it is shown that such operators preserve the hypo-analyticity of a distribution.

1. INTRODUCTION

The main concepts relating to hypo-analyticity were introduced by Baouendi, Chang, and Treves in [1], Chapter 1. We shall summarize some of these in this section.

Suppose Ω is a C^∞ manifold of dimension $m+n$. A hypo-analytic structure on Ω is the data of an open covering (U_α) of Ω and for index α, of m C^∞ functions $Z^1_\alpha, \dots, Z^m_\alpha$ satisfying the following two conditions:

(i) $dZ'_\alpha, \dots, dZ^m_\alpha$ are linearly independent at each point of U_α;
(ii) if $U_\alpha \cap U_\beta \neq \varnothing$, there are open neighborhoods $\mathscr{O}_\alpha$ of $Z_\alpha(U_\alpha \cup U_\beta)$ and $\mathscr{O}_\beta$ of $Z_\beta(U_\alpha \cap U_\beta)$ and a holomorphic map F^α_β of $\mathscr{O}_\alpha$ onto $\mathscr{O}_\beta$, such that

$$Z_\beta = F^\alpha_\beta \circ Z_\alpha \qquad \text{on} \quad U_\alpha \cap U_\beta.$$

When the Z^j are real-valued and $n = 0$, such a structure specializes to a real analytic structure. A distribution h defined in an open neighborhood of a point p_0 of Ω is hypo-analytic at p_0 if there is a hypo-analytic local chart (U_α, Z_α) whose domain contains p_0 and a holomorphic function $\tilde{h}_\alpha$ defined on an open neighborhood of $Z_\alpha(p_0)$ in C^m such that $h = \tilde{h}_\alpha \circ Z$ in a neighborhood of p_0.

By a hypo-analytic local chart we mean an $(m+1)$-tuple $(U, Z^1, \dots, Z^m)$ [abbreviated (U, Z)] consisting of an open subset U of Ω and of m hypo-analytic functions $Z^1, \dots, Z^m$ whose differentials are linearly independent at every point of U.

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In general, the mapping $Z = (Z^1, \dots, Z^m): U \to C^m$ is not a local embedding. However, when $\dim \Omega = m$, this mapping is a local diffeomorphism. Throughout this paper we will assume that the dimension of Ω is m.

2. Preliminaries

We will reason in a hypo-analytic local chart (U, Z) in Ω. We shall assume that the open set U has been contracted sufficiently so that the mapping $Z = (Z^1, \dots, Z^m): U \to C^m$ is a diffeomorphism of U onto $Z(U)$ and that U is the domain of local coordinates x_j $(1 \le j \le m)$ all vanishing at a "central point" which will be denoted by 0. We will suppose $Z(0) = 0$ and denote by Z_x the Jacobian matrix of the Z^j with respect to the x^k. Substitution of $Z_x(0)^{-1}Z(x)$ for $Z(x)$ will allow us to assume that $Z_x(0) =$ the identity matrix. This will permit us to take the real part of the Z^j $(j = 1, \dots, m)$ as coordinates and write in these new coordinates

$$Z^j = x^j + \sqrt{1}\Phi^j(x), \qquad j = 1, \dots, m,$$

where $\Phi = (\Phi^1, \dots, \Phi^m)$ is real-valued whose differential at the origin is 0. Moreover, the functions Z^j are selected so that all the derivatives of order two of the Φ^j vanish at the origin. Indeed if this is not already so it suffices to replace each Z^j by

$$Z^j - \sqrt{-1}/2 \sum\sum \frac{\partial^2 \Phi^j}{\partial x^k \partial x^l}(0) Z^k Z^l.$$

We will use Z_x^* to denote the transpose of the inverse of the matrix Z_x.

Since the first and second derivatives of all the Φ^j are zero at the origin, after contracting U if necessary, we can find a number c, $0 < c < 1$ such that for all $x, y \in U$ and for all $\xi \in R_m$,

$$|\operatorname{Im} Z_x^*(x)\xi| \le c |\operatorname{Re} Z_x^*(x)\xi|$$

and

$$\begin{aligned} \text{(2.1)} \quad \operatorname{Re}\left\{\sqrt{-1} Z_x^*(x)\xi \cdot (Z(x) - Z(y)) - < Z_x^*(x)\xi > (Z(x) - Z(y))^2\right\} \\ \le -c|\xi|\,|Z(x) - Z(y)|^2, \qquad \text{where } \langle \zeta \rangle = (\zeta_1^2 + \cdots + \zeta_m^2)^{1/2}. \end{aligned}$$

3. Hypo-analytic pseudodifferential operators

Our aim is to introduce pseudodifferential operators that are naturally associated with hypo-analytic structures. This definition generalizes analytic pseudodifferential operators (for a treatment of the analytic theory see [6]).

Definition 3.1. Let d be a real number. We denote by $\tilde{S}^d(U, U)$ the space of holomorphic functions $\tilde{a}(z, w, \theta)$ in a product set $\mathscr{O} \times \mathscr{O} \times \mathscr{C}$ with $\mathscr{O}$ an open neighborhood of $Z(U)$ and $\mathscr{C}$ an open cone in $C_m \setminus \{0\}$ containing $R_m \setminus \{0\}$, which have the following property:

Given any compact subset K of $\mathscr{O}$ and any closed cone $\mathscr{C}' \subset \mathscr{C}$ whose interior contains $R_m \setminus \{0\}$, there is a constant $r > 0$ such that for all z, w in K and all θ in $\mathscr{C}'$, we have :

$$|\tilde{a}(z, w, \theta)| \le r(1 + |\theta|)^d.$$

Definition 3.2. We say that a C^∞ function $a(x, y, \theta)$ in $U \times U \times R_m$ is a hypo-analytic amplitude of degree d and we write $a \in S^d(U, U)$ if there is $\tilde{a} \in \tilde{S}^d(U, U)$ such that

$$a(x, y, \theta) = \tilde{a}(Z(x), Z(y), \theta), \qquad \text{for all } x \text{ in } U, \quad y \text{ in } U, \quad 0 \ne \theta \in R_m.$$

Let $a(x, y, \theta) = \tilde{a}(Z(x), Z(y), \theta)$ be a hypo-analytic amplitude of degree $d \in R$ in $U \times U$. For any $\varepsilon > 0$ and $u \in C_c^0(U)$ we define the linear operator

$$A^\varepsilon u(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1}\xi \cdot (Z(x) - Z(y) - \varepsilon|\xi|^2) a(x, y, \xi) u(y)\, dZ(y)\, d\xi.$$

We contract U sufficiently so that for every $x, y \in U$ and $\xi \in R_m$ the point $Z_x^*(x)\xi + \sqrt{-1}\langle Z_x^*(x)\xi\rangle(Z(x) - Z(y))$ will remain in the cone in which $a(x, y, \cdot)$ is defined. We observe that each $A^\varepsilon u$ is a hypo-analytic function.

Theorem 3.1. *When $\varepsilon \to 0$, A^ε converges to a continuous linear operator A: $E'(U) \to \mathscr{D}'(U)$ which maps $C_c^\infty(U)$ into $C^\infty(U)$ continuously.*

Proof. We deform the path of ξ-integration from R_m to the image of R_m under the map

$$\xi \to \zeta(\xi) = Z_x^*(x)\xi + \sqrt{-1}\langle Z_x^*(x)\xi\rangle(Z(x) - Z(y)).$$

Thus

$$\begin{aligned} A^\varepsilon u(x) &= \left(\frac{1}{4\pi^3}\right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1} Z_x^*(x)\xi \cdot (Z(x) - Z(y)) \\ &\qquad - \langle Z_x^*(x)\xi\rangle (Z(x) - Z(y))^2 - \varepsilon\langle\zeta(\xi)\rangle^2) \\ &\qquad\qquad \times a(x, y, \zeta(\xi)) u(y) \det\left(\frac{\partial\zeta}{\partial\xi}\right) dZ(y)\, d\xi. \end{aligned}$$

If the amplitude a has degree $d < -m - 1$ and $u \in C_c(U)$, condition (2.1) will imply that $A^\varepsilon u$ converges uniformly on compact subsets of U to a continuous function Au. Moreover, in this case, $A: C_c^0(U) \to C^0(U)$ will be a continuous operator.

In general, if the degree of $a = d$, we consider the holomorphic functions

$$\tilde{A}^\varepsilon u(z) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1}\xi \cdot (z - Z(y)) - \varepsilon|\xi|^2) \tilde{a}(z, Z(y), \xi) u(y)\, dZ(y)\, d\xi.$$

We denote the Laplacian $\sum_{j=1}^{m} D_{z_j}^2$ by Δ_z and write

$$\tilde{A}^{\varepsilon} u(z) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_U \int_{R_m} (1-\Delta_z)^k \left\{ e^{\sqrt{-1}\xi(z-Z(y))-\varepsilon|\xi|^2} \right\} \frac{\tilde{a}}{(1+|\xi|^2)^k} u(y)\, dZ(y)\, d\xi .$$

In the latter, we apply the transposed Leibniz formula to get

$$\left\{ (1-\Delta_z)^k e^{\sqrt{-1}\xi(z-\omega)} \right\} \tilde{a}(z,\omega,\xi) = \sum_{|\alpha+\beta|\le 2k} c_{\alpha,\beta} \left(\frac{\partial}{\partial z}\right)^{\alpha} \left[e^{\sqrt{-1}\xi(z-\omega)} \left(\frac{\partial}{\partial z}\right)^{\beta} \tilde{a}(z,\omega,\xi) \right],$$

where the $c_{\alpha,\beta}$ are integers.

We can thus write $A^{\varepsilon} u(x) = \sum_{|\alpha|\le 2k} M^{\alpha}(A_{\alpha}^{\varepsilon} u(x))$, where the A_{α}^{ε} are defined like A^{ε} except that their amplitudes have degree $\le d-2k$. For k sufficiently large we have shown that $A^{\varepsilon} u$ converges to a continuous function. Therefore, $A^{\varepsilon} u$ converges to Au in the space $\mathscr{D}'(U)$. Suppose now $u \in \mathscr{E}'(U)$. We choose continuous functions $u_{\alpha} \in C_c^0(U)$ that satisfy $u = \sum_{|\alpha|\le k} M^{\alpha} u_{\alpha}$. We may integrate by parts to get, for each α, $A^{\varepsilon}(M^{\alpha} u_{\alpha}) = A_{\alpha}^{\varepsilon}(u_{\alpha})$, where the degree of the amplitude of A_{α}^{ε} is $\le |\alpha| + d$. Thus $A^{\varepsilon} u = \sum_{|\alpha|\le k} A_{\alpha}^{\varepsilon}(u_{\alpha})$ which brings us to a situation already considered. We conclude that $A^{\varepsilon} u \to Au$ in $\mathscr{D}'(U)$.

Suppose now $u \in C_c^{\infty}(U)$. We denote $\sum_{j=1}^{m} M_j^2$ by Δ_M, where the M_j are vector fields satisfying $M_j Z^k = \delta_j^k$. Integration by parts gives

$$(3.2)\quad A^{\varepsilon} u(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1}\xi \cdot (Z(x)-Z(y)) - \varepsilon|\xi|^2) \times \frac{(1-\Delta_M)^k \{a(x,y,\xi)u(y)\}}{(1+|\xi|^2)^k} dZ(y)\, d\xi.$$

After deforming contour as in (3.1), we see that $A^{\varepsilon} u$ converges in $C_c^0(U)$ to the continuous function Au. Moreover, the same convergence also occurs for $M^{\alpha}(A^{\varepsilon} u)$ for all α. It follows that $Au \in C^{\infty}(U)$. Finally, we will show that the operator $A: C_c^{\infty}(U) \to C_c^{\infty}(U)$ is continuous. Let $u \in C_c^{\infty}(U)$. We can write (3.2) as

$$(3.3)\quad Au(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1} Z_x(x)\xi \cdot (Z(x)-Z(y)) - \langle Z^*(x)\xi\rangle (Z(x)-Z(y))^2) \times \frac{(1-\Delta_M)^k \{a(x,y,\zeta(\xi))u(y)\}}{(1+|\xi|^2)^k} \det \frac{\partial \zeta}{\partial \xi} dZ(y)\, d\xi.$$

We note that both the exponential term and $\det \partial\zeta/\partial\xi$ are bounded. Suppose now the sequence $u_n \to u$ in $C_c^{\infty}(U)$. Then (3.3) shows that $Au_n \to Au$ in

$C_c^0(U)$. To conclude the proof, it suffices to show that for every multi-index α, the sequence $M^\alpha(Au_n)$ is uniformly convergent on compact subsets of U. For each α, there is an amplitude b^α of degree $\leq d + |\alpha|$ such that

$$M^\alpha(A^\varepsilon u)(x)$$
$$= \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m} \int_U \exp(\sqrt{-1}\xi\cdot(Z(x)-Z(y))-\varepsilon|\xi|^2) b^\alpha(x, y, \xi) u(y)\, dZ(y)\, d\xi.$$

By what we have already seen, the right-hand side converges in the space $C_c^0(U)$.

Definition 3.3. The operator $A: \mathscr{E}'(U) \to \mathscr{D}'(U)$ of Theorem 3.1 will be called a hypo-analytic pseudodifferential operator.

Example. A hypo-analytic differential operator on Ω may be defined as a linear differential operator P on Ω satisfying the following property:

For every open subset Ω' of Ω and every hypo-analytic function f on Ω', Pf is hypo-analytic on Ω'. In the hypo-analytic chart (U, Z), let M_j $(1 \leq j \leq m)$ be the vector fields satisfying

$$M_j Z^k = \delta_j^k.$$

Then a hypo-analytic differential operator P takes the form

$$P = \sum_{|\alpha|\leq k} a_\alpha M^\alpha,$$

where each a_α is a hypo-analytic function on U. We will show that such an operator is an example of a hypo-analytic pseudodifferential operator. For $u \in E'(U)$ and $\varepsilon > 0$ let

$$u^\varepsilon(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m} \int_U \exp(\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2) u(y)\, dZ(y)\, d\xi.$$

In [1], the authors observed that $u^\varepsilon \to u$ in the space $\mathscr{D}'(U)$. Write P as $\sum_{|\alpha|\leq k} b_\alpha N^\alpha$ where each b_α is hypo-analytic and $N_j = -\sqrt{-1}M_j$ for each j. We then have

$$N_j u^\varepsilon(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m} \int_U \exp(\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2) \xi_j u(y)\, dZ\, d\xi$$

for each j and therefore

$$Pu^\varepsilon(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m} \int_U \exp(\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2) \times \left(\sum_{|\alpha|\leq k} b_\alpha(x)\xi^\alpha\right) u(y)\, dZ\, d\xi.$$

When $\varepsilon \to 0$, we get $Pu = Au$, where A is the hypo-analytic pseudodifferential operator whose amplitude is

$$\sum_{|\alpha|\leq k} b_\alpha(x)\xi^\alpha.$$

4. Pseudolocal property

The aim of this section is to show that hypo-analytic pseudodifferential operators map hypo-analytic functions to hypo-analytic functions.

Since the first and second derivatives of Φ vanish at the origin, after shrinking U if necessary, we may assume that for all x, y in U,

$$|\Phi(x) - \Phi(y)| \leq |x - y|/2$$

and

$$|\Phi(y)| \leq 1/2|y|^2. \tag{4.1}$$

We shall need the following lemma.

Lemma 4.1. *Let A be a hypo-analytic pseudodifferential operator with amplitude $a(x, y, \xi) = \tilde{a}(Z(x), Z(y), \xi)$ and let u be in $E'(U)$. If u vanishes in some neighborhood of 0, Au is hypo-analytic at 0.*

Proof. For each $\varepsilon > 0$ we consider the holomorphic function

$$\widetilde{A}^\varepsilon u(z) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1}\xi(z - Z(y)) - \varepsilon|\xi|^2)\tilde{a}(z, Z(y), \xi)u(y)\, dZ(y)\, d\xi.$$

We deform the path of ξ-integration from R_m to the image of R_m under the map $\zeta(\xi) = \xi + \sqrt{-1}|\xi|(z - Z(y))$ and write

$$\widetilde{A}^\varepsilon u(z) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1}\xi(z - Z(y)) - |\xi|(z - Z(y))^2 - \varepsilon\langle\zeta(\xi)\rangle)$$
$$\times \tilde{a}(z, Z(y), \zeta(\xi))u(y)\frac{\partial\zeta}{\partial\xi} dZ(y)\, d\xi.$$

Let $Q(z) = \operatorname{Re}\left\{\sqrt{-1}\xi \cdot (z - Z(y)) - |\xi|\,(z - Z(y))^2\right\}$. Using (4.1) we have $Q(0) = \xi \cdot \phi(y) - |\xi|(|y|^2 - |\phi(y)|^2) \leq -\frac{1}{4}|y|^2|\xi|$. Let d be a positive number such that $y \in \operatorname{supp} u \Rightarrow |y| > d$. Then $Q(0) \leq -\frac{1}{4}d^2|\xi|$ which by continuity implies that $Q(z) \leq -\frac{1}{3}d^2|\xi|$ for z in a sufficiently small neighborhood of 0. Therefore, as $\varepsilon \to 0$, $\widetilde{A}^\varepsilon u(z)$ converges uniformly on some neighborhood of 0. It follows that $Au(x) = \widetilde{A}u(Z(x))$ is hypo-analytic at 0.

Theorem 4.2. *Suppose A is a hypo-analytic pseudodifferential operator and $u \in \mathscr{E}'(U)$. If u is hypo-analytic at 0 then Au is hypo-analytic at 0.*

Proof. Let $\tilde{u}$ be a holomorphic function such that $u(y) = \tilde{u}(Z(y))$ for y near 0. In the integral for $\widetilde{A}^{\varepsilon}u(z)$ we deform the "y-contour" from U to the image of U under the map $Z(y) \to \widetilde{Z}(y) = Z(y) - \sqrt{-1}\chi(y)d\xi/|\xi|$, where d is a sufficiently small positive number, $\chi \in C_c^{\infty}(U)$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ near 0 and $\operatorname{supp}\chi$ sufficiently small. We may thus write

$$\widetilde{A}^{\varepsilon}u(z) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m}\int_U \exp(\sqrt{-1}\xi(z - Z(y)) - d\chi(y)|\xi| - \varepsilon|\xi|^2) \\ \times a(z, \widetilde{Z}(y), \xi)\tilde{u}(\widetilde{Z}(y))\, d\widetilde{Z}(y)\, d\xi.$$

We next deform the ξ-integration to the image of R_m under the map $\xi \to \zeta(\xi) = \xi + \sqrt{-1}|\xi|(z - Z(y))$. We will show that $\widetilde{A}^{\varepsilon}u(z)$ converges uniformly near $z = 0$. To prove this, we will estimate

$$Q(z) = \operatorname{Re}\{\sqrt{-1}\xi(z - Z(y)) - |\xi|(z - Z(y))^2 - d\chi(y)\langle\zeta(\xi)\rangle\}.$$

Lemma 4.1 allows us to shrink the support of u so that when $y \in \operatorname{supp} u$ and z is small enough, $|\xi|/2 \leq \operatorname{Re}\langle\zeta(\xi)\rangle$. Moreover, for such z and y we have: $Re\{\sqrt{-1}\xi(z - Z(y)) - |\xi|(z - Z(y))^2\} \leq (-|y|^2/8 + 3|z|)|\xi|)$. Therefore for z near 0, $Q(z) \leq -(|y|^2/8 + d\chi(y)/2 - 3|z|)|\xi|$. This estimate together with the fact that $\chi(y) \equiv 1$ near $y = 0$ yield the required result.

In [1] the authors microlocalized hypo-analyticity by first adapting Sato's definition and then showing its equivalence with the one derived from the Fourier–Bros–Iagolnitzer transform [4]. The operators defined in this paper also preserve microlocal hypo-analyticity [3].

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