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GENERALIZATION OF A CLASSICAL THEOREM OF PÓLYA AND SZEGÖ

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ABSTRACT. An entire function of exponential type bounded on the real axis is bounded along every horizontal line. A generalization of this theorem to the class of entire functions of finite order is given.

1. INTRODUCTION

Entire functions of exponential type have been extensively studied in the past. However, there are not many known results concerning functions of order larger than one. The purpose of this note is to extend a classical result on functions of exponential type due to Pólya and Szegö, to the class of functions of finite order.

In what follows, A, M, τ, c, will be positive constants and $\rho > 1$. For $z \in \mathbf{C}$, we shall write indistinctly $z = x + iy = re^{i\theta}$.

Pólya and Szegö established the following ([7] Vol. 2, p. 33, problem 202; see also Duffin and Schaeffer [2] or Plancherel and Pólya [6]):

Theorem A. *If $f(z)$ is an entire function of exponential type τ such that*

$$|f(x)| \le M \tag{1}$$

for all $x \in \mathbf{R}$, then

$$|f(x+iy)| \le Me^{\tau|y|}.$$

A more general result is given in [1], p. 82, Theorem 6.2.3.

Theorem B. *If $f(z)$ is regular and of exponential type in the first quadrant, $|f(iy)| \le Ae^{\tau y}$ $(0 \le y < \infty)$ and* (1) *is satisfied* $(0 \le x < \infty)$, *then*

$$|f(x+iy)| \le \max(A, M)e^{\tau y} \qquad (0 \le x < \infty, 0 \le y < \infty).$$

In [4] we showed analogous theorems stated for functions of order larger than one. As a matter of fact, we proved

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Theorem C. *If $f(z)$ is regular in the upper half plane* $\operatorname{Im} z \geq 0$ *and such that*

$$|f(z)| \leq Ae^{\tau|z|^\rho} \tag{2}$$

for all z in $\operatorname{Im} z \geq 0$, *and*

$$|f(x)| \leq Me^{-c|x|^\rho} \tag{3}$$

for all $x \in \mathbf{R}$, then for every $c' < c$ we have

$$|f(x+iy)| \leq \max(A, M)\exp(ky^\rho - c'|x|^\rho) \tag{4}$$

for $x \in \mathbf{R}$ and $y \geq 0$, where $k = (\tau + c)^{\rho+1}(2\rho/(c-c'))^\rho$.

Theorem C is not a generalization of Theorem A in the sense that, when $\rho = 1$, the hypothesis (3) is not equivalent to the hypothesis (1).

2. Preliminaries

We define the function $\lambda(\rho)$ by

$$\lambda(\rho) = \rho(1 - 1/\rho)^{1-\rho}. \tag{5}$$

It is easy to see that $\lambda(\rho)$ is bounded by $e\rho$ and $\lambda(\rho) \searrow 1$ as $\rho \to 1^+$. Since $\cos\theta \geq 1 - (2/\pi)\theta$ and $\rho\sin\theta \geq \sin(\rho\theta)$ for $\theta \in [0, \pi/2\rho]$, $\cos\theta \geq 1 - 1/\rho$ in $[0, \pi/2\rho]$. Thus,

$$\sin(\rho\theta)\csc\theta(\sec\theta)^{\rho-1} \leq \lambda(\rho) \tag{6}$$

for all $\theta \in [0, \pi/2\rho]$. If ρ is an integer there is a sharper relation:

Lemma. *For any positive integer n we have*

$$\sin(n\theta)\csc\theta(\sec\theta)^{n-1} \leq n \tag{7}$$

for all θ in $[0, \pi/2n]$.

Proof. We proceed by induction. Consider $\psi_n(\theta) = n\sin\theta(\cos\theta)^{n-1} - \sin(n\theta)$. $\psi_1(\theta) \equiv 0$. Suppose now that for some n, $\psi_n(\theta) \geq 0$ in $[0, \pi/2n]$. By a direct calculation we obtain $\psi_{n+1}(\theta) = \cos\theta\psi_n(\theta) + \sin\theta[\cos^n\theta - \cos(n\theta)]$. It is easy to see that the expression in brackets is positive so that $\psi_{n+1}(\theta) \geq 0$ in $[0, \pi/2n]$ and (7) follows.

3. Main results

Next we extend Theorem A to the class of entire functions of finite order.

Theorem 1. *Let $f(z)$ be regular in the angle* $G = \{z : 0 \leq \theta \leq \pi/2\rho\}$ *and such that* (1) *and* (2) *hold* $(x \geq 0, z \in G)$. *Then for each $y > 0$ we have*

$$|f(x+iy)| \leq \max(A, M)\exp(\tau y\lambda(\rho)x^{\rho-1}) \tag{8}$$

for all $x \geq y\cot(\pi/2\rho)$, where $\lambda(\rho)$ is defined by (5) .

Proof. Let $g(z) = f(z)e^{i\tau z^\rho}$. Here z^ρ denotes the single-valued analytic branch of the multiple-valued function $z^\rho = \exp(\rho\ln z)$ that takes positive values for positive real z. For real x, $|g(x)| = |f(x)| \leq M$, and on the ray $\theta = \pi/2\rho$, we have

$$|g(re^{i\theta})| = |f(re^{i\theta})|e^{-\tau r^{\rho}} \leq A.$$

By the Phragmén–Lindelöf theorem, $|g(re^{i\theta})| \leq \max(A, M)$ throughout G, which leads to

$$|f(re^{i\theta})| \leq \max(A, M)\exp(\tau\sin(\rho\theta)r^{\rho}) \tag{9}$$

for all $z \in G$. Replacing r^{ρ} by $y\csc\theta(\sec\theta)^{\rho-1}x^{\rho-1}$ and applying (6) we obtain (8).

When ρ is a positive integer there is a sharper result.

Theorem 2. *Under the hypotheses of Theorem* 1, *if* ρ *is a positive integer* ($\rho = n$), *then*

$$|f(x+iy)| \leq \max(A, M)\exp(n\tau yx^{n-1})$$

for $x \geq y\cot(\pi/2n)$.

Proof. Proceeding as in the proof of Theorem 1, we obtain (9) with $\rho = n$. Substituting r^{n} by $y\csc\theta(\sec\theta)^{n-1}x^{n-1}$ and applying (7), the conclusion follows.

Theorem 3. *Suppose that* $f(z)$ *is regular in* $G = \{z\colon 0 \leq \theta \leq \pi/2\rho\}$, *satisfies* (2), *and for some* $c > 0$,

$$|f(x)| \leq Me^{-cx^{\rho-1}}$$

for all $x \in \mathbf{R}_{+}$. *Then for each* $0 < k < c/\tau\rho^{2}$, *there is* $B > 0$ *such that* $|f(z)|$ *is bounded by* B *in the region* $G \cap \{0 \leq y \leq k\}$.

Proof. Consider $g(z) = f(z)\exp(cz^{\rho-1})$. Clearly $g(z)$ satisfies (1). Let $\tau' = c/k\rho^{2}$ so that $\tau' > \tau$. There is $A' > 0$ such that $|g(z)| \leq A'\exp(\tau'|z|^{\rho})$, $z \in G$. Proceeding in the same way as in the proof of Theorem 1 we obtain for $g(z)$ the following relation which is analogous to (9)

$$|g(re^{i\theta})| \leq \max(A', M)\exp(\tau'\sin(\theta\rho)r^{\rho}).$$

Hence,

$$\begin{aligned}|f(re^{i\theta})| &\leq \max(A', M)\exp(\tau'\sin(\theta\rho)r^{\rho} - c\cos(\theta(\rho-1))r^{\rho-1}) \\ &= \max(A', M)\exp\left[\left(\tau' y\frac{\sin(\theta\rho)}{\sin\theta} - c\cos(\theta(\rho-1))\right)\left(\frac{x}{\cos\theta}\right)^{\rho-1}\right].\end{aligned}$$

If $y \leq k < c/\tau\rho^{2}$, since $\rho\sin\theta \geq \sin(\rho\theta)$ for $0 \leq \theta \leq \pi/2\rho$,

$$\begin{aligned}\tau' y\frac{\sin(\theta\rho)}{\sin\theta} - c\cos(\theta(\rho-1)) &\leq c(\rho^{-1} - \cos(\theta(\rho-1))) \\ &\leq c\left(\rho^{-1} - \cos\left(\frac{\pi}{2\rho}(\rho-1)\right)\right).\end{aligned}$$

Using the estimate $\cos u \geq 1 - (2/\pi)u \quad 0 \leq u \leq \pi/2$, the last expression is less or equal to 0. Letting $B = \max(A', M)$ the proof is achieved.

Corollary. *Under the hypotheses of Theorem* 3, $|f(z)| \to 0$ *as* $|z| \to \infty$ *uniformly in any strip* $0 \leq y \leq k < c/\tau\rho^2$.

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