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STABLE ISOMORPHISM OF HEREDITARY C^*-SUBALGEBRAS AND STABLE EQUIVALENCE OF OPEN PROJECTIONS

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ABSTRACT. We relate the stable isomorphism of two hereditary C^*-subalgebras to the stable equivalence of the corresponding open projections. We prove that if A is completely σ-unital, then $\mathrm{her}(p)$ and $\mathrm{her}(q)$ generate the same closed ideal of A iff $p\otimes 1 \sim q\otimes 1$ in $(A\otimes K)^{**}$ iff the central supports of p and q in A^{**} are the same. If, in addition, $p \perp q$, then the above three equivalent conditions are equivalent to the condition: $p \otimes 1$ and $q \otimes 1$ are in the same path component of *open* projections in $(A \otimes K)^{**}$.

0. INTRODUCTION

Let A be any C^*-algebra. We denote the multiplier algebra of A by $M(A)$ and the Banach space double dual of A by A^{**}. The Murray-von Neumann equivalence of projections in A^{**} is denoted by $\sim$. The set of hereditary C^*-subalgebras of A is denoted by $H(A)$ and the set of closed ideals of A by $I(A)$. It is clear that $I(A) \subset H(A)$. Strong Morita equivalence is an equivalence relation on $H(A)$ (see [10] and [4]). We denote the set of strong Morita equivalence classes of $H(A)$ by $\widetilde{H}(A)$ and the set of strong Morita equivalence classes of $I(A)$ by $\widetilde{I}(A)$. If A_0 is a hereditary C^*-subalgebra of A, we denote the strong Morita equivalence class containing A_0 by $[A_0]$.

We say that a C^*-algebra A is completely σ-unital if every hereditary C^*-subalgebra of A is σ-unital. It is clear that any separable C^*-algebra is completely σ-unital.

It was proved in [5] that two σ-unital C^*-algebras A and B are strongly Morita equivalent iff A and B are stably isomorphic (i.e. $A\otimes K \simeq B\otimes K$). Hence for a completely σ-unital C^*-algebra A each class $[A_0]$ coincides with the equivalence class $[A_0]_s$ in the sense of stable isomorphism. It follows easily from the results of [5] and [3, 2.5] that if A is σ-unital, then there is a bijection between $\widetilde{I}(A)$ and $\widetilde{H}(A)$.

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It is well known that $A^{**} \ni p \leftrightarrow pA^{**}p \cap A \in H(A)$ is a bijection between the set of open projections and the set of all hereditary C^*-subalgebras of A. Moreover central open projections correspond to closed ideals under the bijection (see [2]). Here $\mathrm{her}(p)$ denotes the hereditary C^*-subalgebra of A corresponding to an open projection p in A^{**}.

Two open projections in A^{**} are said to be stably equivalent if there is a $v \in (A \otimes K)^{**}$ such that $vv^* = p \otimes 1$ and $v^*v = q \otimes 1$. It is clear that the stable equivalence of open projections is an equivalence relation on the set of open projections in A^{**}. Let $\widetilde{O}(A)$ be the set of equivalence classes of open projections in A^{**} under stable equivalence.

In this short note we shall show that there is a bijection $I(A) \leftrightarrow \widetilde{O}(A)$ and describe the stable isomorphism of hereditary C^*-subalgebras of A by the stable equivalence of the corresponding open projections. Moreover we shall prove that two mutually orthogonal hereditary C^*-subalgebras $\mathrm{her}(p)$ and $\mathrm{her}(q)$ are stably isomorphic iff $\mathrm{her}(p) \otimes K$ can be continuously deformed through hereditary C^*-subalgebras of $A \otimes K$ to $\mathrm{her}(q) \otimes K$. In particular, if two σ-unital C^*-algebras A and B are strongly Morita equivalent (or equivalently stably isomorphic) and

$$L = \begin{pmatrix} A & X \\ \widetilde{X} & B \end{pmatrix}$$

is the linking algebra constructed in [5], then $A \otimes K$ and $B \otimes K$ are path connected by mutually stably isomorphic hereditary C^*-subalgebras of $L \otimes K$. We think that this gives a more basic understanding of the stable isomorphism of hereditary C^*-subalgebras.

We call a hereditary C^*-subalgebra $\mathrm{her}(p)$ of A essential if there is no nonzero hereditary C^*-subalgebra $\mathrm{her}(q)$ such that $xy = 0$ for all x in $\mathrm{her}(p)$, y in $\mathrm{her}(q)$. It is equivalent that there be no nonzero open projection q such that $q \perp p$. Here $\perp$ means orthogonal.

1. Main result

1. **Theorem.** *If A is a completely σ-unital C^*-algebra and p, q are two open projections in A^{**}, then the following are equivalent:*

(a) $\mathrm{her}(p)$ *and* $\mathrm{her}(q)$ *generate the same closed ideal of* A.
(b) *The central supports of p and q in A^{**} are the same.*
(c) *p and q are stably equivalent.*

If, *in addition, $p \perp q$, then the above three conditions are equivalent to:*

(d) *$p \otimes 1$ and $q \otimes 1$ are in the same path component of open projections in* $(A \otimes K)^{**}$.

Consequently any one of the conditions (a)–(d) *implies that* $\mathrm{her}(p) \otimes K \simeq \mathrm{her}(q) \otimes K$.

Proof. (a) $\Leftrightarrow$ (b): Let $I(p)$ be the ideal of A generated by $\mathrm{her}(p)$ for any open projection p. It suffices to show that the central open projection corresponding to $I(p)$ is the central cover $c(p)$ of p in A^{**}.

Let q_0 be the central open projection corresponding to $I(p)$. First of all, $p \leq q_0$ is clear since $\text{her}(p) \subset I(p)$. We need only show that $q_0 \leq c(p)$. Let $A_1 = c(p)A^{**} \cap A$. Then A_1 is a closed ideal of A. Let q_1 be the central open projection corresponding to A_1. Then $q_1 \leq c(p)$. Since $pA^{**}p \subset c(p)A^{**}c(p)$, $\text{her}(p) \subset A_1$. It follows that $I(p) \subset A_1$ and so $q_0 \leq q_1$. Thus $q_0 \leq c(p)$.

(b) $\Rightarrow$ (c): We show that p and $c(p)$ are stably equivalent. Then similarly q and $c(p)$ are stably equivalent, and so p and q are stably equivalent.

Let $I(p)$ be the closed ideal generated by $\text{her}(p)$. As above $I(p) = c(p)A^{**} \cap A$. By hypothesis $\text{her}(p)$ and $I(p)$ are both σ-unital. By Lemma (6.2) in [9], there is a sequence $\{a_i\} \subset I(p)$ such that $a_i a_i^* \in \text{her}(p)$ for all $i \geq 1$ and $\sum_{i=1}^{\infty} a_i^* a_i = c(p)$ with convergence in the strict topology in $M(I(p))$.

Define $u = \sum_{i=1}^{\infty} a_i \otimes e_{i1}$, then it is routine to check that the sum converges in the strict topology in $M(I(p) \otimes K)$ and $u^*u = c(p) \otimes e_{11}$ and $uu^* \leq p \otimes 1$ (see the proof of Lemma (2.4) of [3]), where we need the fact that $a_i a_i^* \in \text{her}(p)$ implies $a_i^* p a_i = a_i^* a_i$ for any $i \geq 1$. Since $M(I(p) \otimes K) \subset (I(p) \otimes K)^{**} \subset (A \otimes K)^{**}$ and the strict topology of $M(I(p) \otimes K)$ is stronger than the relative w^*-topology induced by the one of $(I(p) \otimes K)^{**}$, $u = \sum_{i=1}^{\infty} a_i \otimes e_{i1}$ converges in the w^*-topology of $(I(p) \otimes K)^{**}$ and so it converges in the w^*-topology of $(A \otimes K)^{**}$. Now the proof of Lemma (2.5) of [3] can be repeated to obtain a partial isometry $v \in (A \otimes K)^{**}$ such that $v^*v = c(p) \otimes 1$ and $vv^* = p \otimes 1$.

(c) $\Rightarrow$ (b): Let $c(p \otimes 1)$ and $c(q \otimes 1)$ be the central supports of $p \otimes 1$ and $q \otimes 1$ in $(A \otimes K)^{**}$, respectively. Since $p \otimes 1 \sim q \otimes 1$, $c(p \otimes 1) = c(q \otimes 1)$ by [7, 6.2.8]. It is easy to check that $c(p \otimes 1) = c(p) \otimes 1$ and $c(q \otimes 1) = c(q) \otimes 1$. It follows that $c(p) = c(q)$.

(a) implies that $\text{her}(p) \otimes K \simeq \text{her}(q) \otimes K$ since $\text{her}(p)$ and $\text{her}(q)$ are σ-unital and strongly Morita equivalent ([3, 2.5]).

(d) $\Rightarrow$ (c) since $p \otimes 1$ and $q \otimes 1$ are unitarily equivalent. It is a well known fact that two projections in a C^*-algebra are unitarily equivalent if the distance (in norm) between them is less than one.

(a) $\Rightarrow$ (d): Since $\text{her}(p)$ and $\text{her}(q)$ generate the same closed ideal I, $\text{her}(p)$ and $\text{her}(q)$ are strongly Morita equivalent. Let $X = [\text{her}(p) I \,\text{her}(q)]^- = [\text{her}(p) A \,\text{her}(q)]^-$. Then $X \subset A$ and X is a $\text{her}(p) - \text{her}(q)$-imprimitivity bimodule. Let

$$L = \begin{pmatrix} \text{her}(p) & X \\ \widetilde{X} & \text{her}(q) \end{pmatrix}$$

be the linking algebra of [5]. Since $p \perp q$, L can be identified with a subalgebra of A and so $L \otimes K$ can be identified with a subalgebra of $A \otimes K$. Consequently $M(L \otimes K)$ can be identified with a subalgebra of $(A \otimes K)^{**}$. We assume that $L \otimes K \subset M(L \otimes K) \subset (A \otimes K)^{**}$ from now on. By [3, 2.5], there is $v \in M(L \otimes K)$ such that $vv^* = p \otimes 1$ and $v^*v = q \otimes 1$. Let $u = v + v^*$. Then $u = u^*$, $u^2 = (p+q) \otimes 1$ and $u(p \otimes 1)u = q \otimes 1$. Define a path of unitaries in

$[(p+q)\otimes 1](A\otimes K)^{**}[(p+q)\otimes 1]$ by

$$u(t)=\tfrac{1}{2}(1+e^{it\pi})(p+q)\otimes 1+\tfrac{1}{2}(1-e^{it\pi})u: \qquad 0\leq t\leq 1.$$

Then define a path of projections in $(A\otimes K)^{**}$ by

$$p(t)=u(t)^{*}(p\otimes 1)u(t): \qquad 0\leq t\leq 1.$$

It is easy to check that $p(0)=p\otimes 1$ and $p(1)=q\otimes 1$. It remains to show that $p(t)$ is an open projection for each $t\in[0,1]$. Since p is open, there is a net of positive elements in A such that $a_\lambda \nearrow p$ ([2] and [8]). Let $f_n=\sum_{i=1}^{n} e_{ii}$ for each n. It is obvious that $u(t)^{*}[a_\lambda\otimes f_n]u(t)\nearrow p(t)$ in the w^{*}-topology of $(A\otimes K)^{**}$, where the directed set $\Lambda\times N$ has the dictionary order.

It is sufficient to show $u(t)^{*}[a_\lambda\otimes f_n]u(t)\in A\otimes K$ for each $t\in[0,1]$. By construction it suffices to show that $u[a_\lambda\otimes f_n]$, $[a_\lambda\otimes f_n]u$, and $u[a_\lambda\otimes f_n]u$ are all in $A\otimes K$. In fact, since $a_\lambda\otimes f_n\in(p\otimes 1)(A\otimes K)(p\otimes 1)$ and $v\in M(L\otimes K)$, we obtain that

$$u[a_\lambda\otimes f_n]=v^{*}[a_\lambda\otimes f_n],\ [a_\lambda\otimes f_n]u=[a_\lambda\otimes f_n]v, \text{ and } u[a_\lambda\otimes f_n]u=v^{*}[a_\lambda\otimes f_n]v$$

are all in $L\otimes K\subset A\otimes K$. □

2. *Remarks.* (1) In Theorem 1 (a)⇒(d), if p and q are in $M(A)$, then the path of projections between $p\otimes 1$ and $q\otimes 1$ can be chosen in $M(A\otimes K)$ by the same proof.

(2) An easy consequence of Theorem 1 (a)⇔(b) is that a central projection r in A^{**} is open iff $r=c(p)$ for some open projection p in A^{**}.

(3) Note that the assumption $p\perp q$ is not needed in proving (d)⇒(c), but for (a)⇒(d) this assumption cannot be removed in general. In fact, if p corresponds to a full hereditary C^*-subalgebra of A, then by Theroem (2.5) of [3] her(p) generates A as a closed ideal, but $p\otimes 1$ and $1\otimes 1$ cannot be connected by a path of open projections in $(A\otimes K)^{**}$.

2. Corollaries

3. **Corollary.** *If A is a completely σ-unital simple C^*-algebra,, and* her(p), her(q) *are nonessential hereditary C^*-subalgebras of A, then $p\otimes 1$ and $q\otimes 1$ are in the same path component of open projections in $(A\otimes K)^{**}$ whenever $pq=qp$.*

Proof. By Theorem 1, we may assume $r=pq\neq 0$. Since $pq=qp=r$, r is the open projection corresponding to her$(p)\cap$her(q) by [1]. Since p and q are both nonessential, we can choose nonzero open projections $p_1\perp p$ and $q_1\perp q$. Then we have $p_1\perp r$ and $q_1\perp r$. Now Theorem 1 applies. $p\otimes 1$ is path connected to $p_1\otimes 1$, $p_1\otimes 1$ is path connected to $r\otimes 1$, $r\otimes 1$ is path connected to $q_1\otimes 1$ and $q_1\otimes 1$ is path connected to $q\otimes 1$. Here all paths are in the set of open projections in $(A\otimes K)^{**}$. Therefore $p\otimes 1$ and $q\otimes 1$ are in the same path component of open projections in $(A\otimes K)^{**}$. □

In [6], it was proved that all proper projections in $M(A\otimes K)$ are in the same path component of projections if A is a σ-unital C^*-algebra, where a projection P in $M(A\otimes K)$ is proper iff $P\sim 1\sim 1-P$. With the aid of their result we prove the following:

4. **Corollary.** *If A is a completely σ-unital simple C^*-algebra such that every hereditary C^*-subalgebra contains a nonzero projection then $p\otimes 1$ and $q\otimes 1$ are in the same path component of open projections in $(A\otimes K)^{**}$ whenever p and q are nonzero nonessential open projections in A^{**}.*

Proof. Since p and q are nonessential open projections in A^{**}, there are nonzero open projections $p_0\perp p$ and $q_0\perp q$. By hypotheses, there are nonzero projections p_1 in her(p) and q_1 in her(q). By Theorem 1 $p\otimes 1$ and $p_0\otimes 1$ can be joined by a path of open projections in $(A\otimes K)^{**}$ and also $q\otimes 1$ and $q_0\otimes 1$ can be joined by a path of open projections in $(A\otimes K)^{**}$. Similarly, $p_0\otimes 1$ and $p_1\otimes 1$ are in the same path component of open projections of $(A\otimes K)^{**}$, and so are $q_0\otimes 1$ and $q_1\otimes 1$. Since A is simple, $A\otimes K$ is simple. Thus $p_1\otimes 1$ and $q_1\otimes 1$ are both proper by Theorem (2.5) of [3]. By [6, Lemma 1] $p_1\otimes 1$ and $q_1\otimes 1$ can be joined by a path of projections in $M(A\otimes K)$. Therefore $p\otimes 1$ and $q\otimes 1$ are in the same path component of open projections in $(A\otimes K)^{**}$. □

5. *Question.* If A is a completely σ-unital C^*-algebra and her(p) and her(q) generate the same closed ideal of A, and $\|pq\|<1$, does it follow that $p\otimes 1$ and $q\otimes 1$ are in the same path component of open projections in $(A\otimes K)^{**}$?

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References

1. C. A. Akemann, *The general Stone-Weierstrass problem*, J. Funct. Anal. **4** (1969), 277–294.
2. ——, *Left ideal structure of C^*-algebras*, J. Funct. Anal. **6** (1970), 305–317.
3. L. G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** (1977), 335–348.
4. ——, *Close hereditary C^*-subalgebras and the structure of quasi-multipliers* (preprint).
5. L. G. Brown, P. Green and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras*, Pacific J. Math. **71** (1977), 349–363.
6. J. Cuntz and Nigel Higson, *Kuiper's theorem for Hilbert modules*, Operator Alg. and Math. Phys. A.M.S., Vol. 62, Proc. of a Summer conference June 17–21, 1985.
7. R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebra*, Vol. I, II. Academic Press, New York, 1983–1986.
8. G. K. Pedersen, *Applications of weak * semicontinuity in C^*-algebra theory*, Duke Math. J. **39** (1972), 431–450.

9. D. Olsen and G. K. Pedersen, *Applications of the Connes spectrum to C^*-dynamical systems* III, J. Funct. Anal. **45** (1982), 357–390.

10. M. A. Rieffel, *Induced representations of C^*-algebras*, Adv. in Math. **13** (1974), 176–257.

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