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MONOTONE CLOSURES OF COMMUTATIVE C^*-ALGEBRAS

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ABSTRACT. We show by an example that an analogue of Pedersen's up-down-up theorem does not hold for monotone complete C^*-algebras.

1. INTRODUCTION

Let B be a von Neumann algebra and A a C^*-subalgebra of B which generates B as a von Neumann algebra (or equivalently, is strongly dense in B). For a subset S of a self-adjoint part B_{sa} of B let S^m (resp., S_m) denote the set of elements in B_{sa} which are obtained as the strong limits of monotone increasing (resp., decreasing) nets from S. Kadison showed in [4] that B_{sa} is itself a unique real linear subspace S of B_{sa} such that $A_{sa} \subset S$ and $S^m = S$, and asked the question whether $B_{sa} = (\cdots(((A_{sa})^m)_m)^m \cdots)_m$ (finitely many steps). As an answer to this question, Pedersen proved in [5] that $B_{sa} = (((A_{sa})^m)_m)^m$ (the up–down–up theorem) in general.

A similar question arises when B is a monotone complete C^*-algebra which is generated by its C^*-subalgebra A as a monotone complete C^*-algebra. However we provide in the following an example of commutative B for which $B_{sa} \neq (\cdots(((A_{sa})^m)_m)^m \cdots)_m$ at all finitely many steps. The existence of such a B is an immediate consequence of the result by Gaifman [2] and Hales [3] on complete Boolean algebras (see also [6]).

Here a C^*-algebra B is called *monotone complete* if every bounded increasing net in B_{sa} has a supremum in the partially ordered set B_{sa}, and a C^*-subalgebra A of B is said to *generate* B *as a monotone complete* C^**-algebra* (or B is the *monotone closure* of A) if B is the only monotone closed C^*-subalgebra C of B (i.e. $(C_{sa})^m = C_{sa}$) containing A.

Henceforth we consider only commutative C^*-algebras, for which monotone completeness is equivalent to being AW^*, and so a C^*-subalgebra of a commutative AW^*-algebra is monotone closed if and only if it is an AW^*-subalgebra. For basic facts on AW^*-algebras see [1].

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2. The example

Following Solovay [6] we construct a complete Boolean algebra. Such a construction is a simplified version of a result of Gaifman [2] and Hales [3]. Let D be any set equipped with the discrete topology, whose cardinality will be specified below, and let X be the product space $D^{\mathbf{N}}$ with the product topology. Then the complete Boolean algebra, $RO(X)$, consisting of all regular open subsets of X is countably generated, [6]. In terms of commutative AW^*-algebras, this fact is rephrased as follows. If B is the commutative AW^*-algebra whose projection lattice coincides with $RO(X)$ (i.e. the spectrum of B is the Stone representation space of $RO(X)$), then a separable C^*-subalgebra A of B, which is generated by some countable projections as a C^*-algebra, generates B as a monotone complete C^*-algebra.

Let define cardinalities κ_n $(n = 0, 1, \dots)$ successively by $\kappa_0 = \aleph_0$, $\kappa_{n+1} = 2^{\kappa_n}$ $(n = 0, 1, \dots)$, and set $\lambda = \sup \kappa_n$.

In the above notation we show the following.

Theorem. *If the cardinality of D is more than λ, then we have*

$$B_{sa} \neq (\dots(((A_{sa})^m)_m)^m \dots)_m$$

at all finitely many steps.

We begin with the next lemma.

Lemma. *Let B be a commutative AW^*-algebra and A a C^*-subalgebra, containing the unit, of B which is generated by its projections A_p as a C^*-algebra. Let $A_1 = C^*((A_{sa})^m)$ denote the C^*-subalgebra of B generated by $(A_{sa})^m$. Then A_1 is generated by projections and we have*

$$\mathrm{card}((A_1)_p) \leq 2^{\mathrm{card}(A_p)},$$

where $\mathrm{card}(E)$ *denotes the cardinality of a set E.*

Proof. Write b^+ and b^- for the positive and negative part of an element b in B_{sa}, so that $b = b^+ - b^-$. If $\{a_i\}$ is an increasing net in A_{sa} with $\sup a_i = x$ in B_{sa}, then $\sup a_i^+ = x^+$ and the support projection of x^+ is the supremum of the support projections of a_i^+. Indeed, as B is commutative, the net $\{a_i^+\}$ is increasing and $\sup a_i^+ \leq x^+$, and similarly, $\inf a_i^- \geq x^-$. Hence $x = \sup a_i \leq \sup a_i^+ + \sup(-a_i^-) = \sup a_i^+ - \inf a_i^- \leq x^+ - x^- = x$, and $x^+ = \sup a_i^+$. Let E be the set of projections in B which are suprema of some subsets of A_p. Then $E \subset (A_{sa})^m$, and the above argument implies that for every $x \in (A_{sa})^m$ and $\mu \in \mathbf{R}$, the support projection of $(x - \mu)^+$ belongs to E, since $x - \mu \in (A_{sa})^m$ and A is generated by A_p. As x is the norm limit of linear combinations of the support projections of $(x - \mu)^+$, $\mu \in \mathbf{R}$, we have $x \in C^*(E)$, and so $A_1 = C^*((A_{sa})^m) = C^*(E)$. Thus $(A_1)_p$ is the Boolean algebra generated by E, and clearly $\mathrm{card}((A_1)_p) = \mathrm{card}(E) \leq 2^{\mathrm{card}(A_p)}$.

Proof of Theorem. With A and B as in the theorem, define C^*-subalgebras A_n $(n = 0, 1, \dots)$ of B inductively by $A_0 = A$, $A_{n+1} = C^*(((A_n)_{sa})^m)$ $(n = 0, 1, \dots)$. Clearly $\mathrm{card}(A_p) = \aleph_0$, and it follows from the lemma that $\mathrm{card}((A_1)_p) \leq 2^{\aleph_0} = \kappa_1, \dots, \mathrm{card}((A_{n+1})_p) \leq 2^{\mathrm{card}((A_n)_p)} \leq 2^{\kappa_n} = \kappa_{n+1}, \dots$. Moreover $(A_{sa})^m \subset C^*((A_{sa})^m) = A_1$, $((A_{sa})^m)_m = (-(A_{sa})^m)^m \subset C^*(((A_1)_{sa})^m) = A_2, \dots$ and so on, and the cardinality of the set of projections in $(\dots(((A_{sa})^m)_m)^m \dots)_m$ (at n steps) is less than or equal to κ_n. On the other hand, $\mathrm{card}(B_p) = \mathrm{card}(RO(X)) \geq \mathrm{card}(D) > \lambda$, and the proof is complete.

Remark. The spectrum of the algebra A in the theorem is homeomorphic to the Cantor set (the totally disconnected, compact metric space without isolated points), as follows from the fact that A is generated by countable projections and has no minimal projections. Indeed, if A has a minimal projection p, then the monotone closure of $A = \mathbb{C}p + (1-p)A$ is $B = \mathbb{C}p + (1-p)B$. This shows that p is also minimal in B, which is clearly impossible. Hence the isomorphism class of A does not depend on the cardinality of D, and the Gaifman–Hales–Solovay theorem stated above tells us that the monotone closure of A can be an arbitrarily large commutative AW^*-algebra when A is embedded in a monotone complete C^*-algebra as a C^*-subalgebra.

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