PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 4, April 1989

A THEOREM ON FUNCTION SPACES

JAN BAARS, JOOST DE GROOT AND JAN VAN MILL

(Communicated by Dennis K. Burke)

ABSTRACT. Let X and Y be normal and first countable spaces, such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Suppose $X^{(\alpha)}$ is countably compact for some $\alpha < \omega_1$. We prove that if $\alpha = 1$ then $Y^{(\alpha)}$ is also countably compact. The first countability condition in this result is essential. We also present examples that if α is not a prime component, then $Y^{(\alpha)}$ need not to be countably compact.

0. INTRODUCTION

Let X and Y be Tychonov spaces. By $C(X)$ we denote the set of all realvalued continuous functions on X. We endow $C(X)$ with a topological vectorspace-structure by considering it to be a subspace of $\mathbb{R}^X$. With this topology we denote $C(X)$ by $C_p(X)$.

In [1] Arhangelskiĭ proved that if $C_p(X)$ is linearly homeomorphic to $C_p(Y)$, and X is compact, then Y is compact. In addition, if X is pseudocompact then Y is pseudocompact. This means in particular that if X and Y are normal then X is countably compact if and only if Y is countably compact. In this note we prove that if X and Y are both normal and first countable such that $C_p(X)$ is linearly homeomorphic to $C_p(Y)$, then $X^{(1)}$ is countably compact if and only if $Y^{(1)}$ is countably compact ($X^{(1)}$ is the set of accumulation points of X). Our technique is inspired by Arhangelskiĭ [1] and Baars, de Groot, van Mill and Pelant [3]. We give two examples showing that our result is "best possible". There exist a first countable normal space X and a normal space Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic but $X^{(1)}$ is not countably compact and $Y^{(1)}$ is countably compact. In addition, there exist two metric spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic but $X^{(2)}$ is compact while $Y^{(2)}$ is not compact ($X^{(2)}$ is the second derivative of X).

Received by the editors April 28, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C35, 57N20.
Key words and phrases. Function spaces.

1. Preliminaries

In this section we give some results from Baars and de Groot [2], and results and definitions from Arhangelskiĭ [1], which we use in section 2.

Let X be a toplogical space and A a subset of X. Let $Y = Y_{X,A}$ be the quotientspace obtained from X by identifying A to one point, say ∞. Let $C_{p,A}(X)$ be the subspace of $C_p(X)$ consisting of those functions which vanish on A, and let $C_{p,0}(Y)$ be the subspace of $C_p(Y)$ consisting of those functions which are zero at ∞.

If two linear spaces X and Y are linearly homeomorphic then we denote that by $X \sim Y$.

1.1 **Lemma** [2]. *Let X be a space and A a subset of X. Then $C_{p,A}(X) \sim C_{p,0}(Y)$.*

For a topological space X we define for every ordinal α the *α-th derivative* $X^{(\alpha)}$ by transfinite induction as follows: (see [5])

(a) $X^{(0)} = X$ and $X^{(1)} = \{x \in X | x$ is an accumulation point of $X\}$.

(b) If α is a successor, say $\alpha = \beta + 1$, then $X^{(\alpha)} = (X^{\beta)})^{(1)}$.

(c) If α is a limit ordinal then $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$.

An ordinal α is a *prime component* whenever for all ordinals β and δ with $\alpha = \beta + \delta$ we have $\delta = 0$ or $\delta = \alpha$. For every ordinal α denote by α' the largest prime component which is less than or equal to α.

By $C_{p,0}([1,\alpha])$ we mean the subspace of $C_p([1,\alpha])$ consisting of those functions which are zero at α.

The next lemma and theorem can be found in [2].

1.2 **Lemma.** *Let α be an ordinal. Then $C_{p,0}([1,\alpha]) \sim C_p([1,\alpha])$.*

1.3 **Theorem.** *Let $\omega \leqq \alpha$, $\beta < \omega_1$. Then $C_p([1,\alpha]) \sim C_p([1,\beta])$ iff $\alpha \leqq \beta < \alpha^{\omega}$.*

The following definitions can be found in [1]. Let X and Y be Tychonov spaces, and $\phi: C(X) \to C(Y)$ a linear mapping. For every $y \in Y$, the *support* of y in X is defined to be the set $\mathrm{supp}(y)$ of all $x \in X$ satisfying the condition that for every neighborhood U of x, there is an $f \in C(X)$ such that $f(X \backslash U) = \{0\}$ and $\phi(f)(y) \neq 0$. For a subset A of Y, we denote $\bigcup_{y\in A} \mathrm{supp}(y)$ by $\mathrm{supp}\, A$. Furthermore ϕ is said to be *effective* if for every f, $g \in C(X)$ and $y \in Y$, such that f and g coincide on a neighborhood of $\mathrm{supp}(y)$, $\phi(f)(y) = \phi(g)(y)$.

A subset A of X is said to be *bounded* if for every $f \in C(X)$, $f(A)$ is bounded in $\mathbb{R}$.

1.4 **Proposition.** ([1] *Arhangelskiĭ*). *Let X and Y be Tychonov spaces and $\phi: C_p(X) \to C_p(Y)$ a linear homeomorphism. Then*

(a) *ϕ is effective,*

(b) *if A is a bounded subset of Y, then* $\operatorname{supp} A$ *is bounded in* X.

For details about ordinals we refer to [5] and [6].

2. Function spaces

In this section we prove the results, announced in the Introduction.

2.1 **Lemma.** *Let X and Y be Tychonov spaces and $\phi: C_p(X) \to C_p(Y)$ a homeomorphism. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C_p(X)$ such that f_n converges pointwise to a discontinuous function $f \in \mathbb{R}^X$. Suppose $g: Y \to \mathbb{R}$ is an accumulation point of the set $\{\phi(f_n) \mid n \in \mathbb{N}\}$. Then g is not continuous.*

Proof. Since $\{f_n \mid n \in \mathbb{N}\}$ is closed and discrete in $C_p(X)$ we have $\{\phi(f_n) \mid n \in \mathbb{N}\}$ is closed and discrete in $C_p(Y)$. □

2.2 **Theorem.** *Let X and Y be topological spaces which are both normal and first countable and let $C_p(X)$ and $C_p(Y)$ be linearly homeomorphic. Then $X^{(1)}$ is countably compact if and only if $Y^{(1)}$ is countably compact.*

Proof. Suppose $X^{(1)}$ is not countably compact and $Y^{(1)}$ is countably compact. Since $X^{(1)}$ is not sequentially compact, there exists a closed discrete set $F = \{x_n \mid n \in \mathbb{N}\}$ in $X^{(1)}$. For every $n \in \mathbb{N}$ let $\{U_j^n \mid j \in \mathbb{N}\}$ be a decreasing open base at x_n and f_j^n a Urysohn function such that $f_j^n(x_n) = 1$ and $f_j^n(X \backslash U_j^n) = 0$. Then $f_j^n \to \chi_{x_n}$ pointwise, where χ_{x_n} is the characteristic function of x_n. Notice that χ_{x_n} is discontinuous. Furthermore let $\phi: C_p(X) \to C_p(Y)$ be a linear homeomorphism and let $g_j^n = \phi(f_j^n)$.

Claim. For every $y \in Y$ and $n \in \mathbb{N}$, the set $\{g_j^n(y) \mid j \in \mathbb{N}\}$ is bounded in $\mathbb{R}$.

Suppose not. Then there are $y \in Y$ and $n \in \mathbb{N}$, such that without loss of generality for every $k \in \mathbb{N}$ there is $j_k \in \mathbb{N}$, with $g_{j_k}^n(y) \geqq 2^k$. The function $f = \sum_{k=1}^{\infty} 2^{-k} f_{j_k}^n \in C_p(X)$, so $\phi(f) = \sum_{k=1}^{\infty} 2^{-k} g_{j_k}^n \in C_p(Y)$. But then we have a contradiction since $\phi(f)(y) = \sum_{k=1}^{\infty} 2^{-k} g_{j_k}^n(y) = \infty$.

For every $y \in Y$, let A_y be compact in $\mathbb{R}$ such that $\{g_j^n(y) \mid j \in \mathbb{N}\} \subset A_y$. Then $\prod_{y\in Y} A_y$ is a compact subset of $\mathbb{R}^Y$. Since $\{g_j^n \mid j \in \mathbb{N}\} \subset \prod_{y\in Y} A_y$, $\{g_j^n \mid j \in \mathbb{N}\}$ has an accumulation point σ_n. By Lemma 2.1, σ_n is discontinuous, say at y_n. Notice that $y_n \in Y^{(1)}$. Since $Y^{(1)}$ is sequentially compact, without loss of generality we may assume that there is $y \in Y$ such that $y_n \to y$. Let $\{V_n \mid n \in \mathbb{N}\}$ be a decreasing open base at y. Without loss of generality $y_n \in V_n$.

Since Y is first countable, for every $n \in \mathbb{N}$ there is a sequence $(y_k^n)_k$ in V_n such that $y_k^n \to y_n$ and

$$(*) \qquad \sigma_n(y_k^n) \nrightarrow \sigma_n(y_n).$$

Let $K = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{y_n, y_k^n\} \cup \{y\}$. Then K is compact. Indeed, let $\mathscr{V}$ be an open cover of K. There is $V \in \mathscr{V}$ with $y \in V$. There is $n_0 \in \mathbb{N}$ such that $y \in V_{n_0} \subset V$. Then $\bigcup_{n \geqq n_0} \bigcup_{k \in \mathbb{N}} \{y_n, y_k^n\} \cup \{y\} \subset V$. Since $\bigcup_{n<n_0} \bigcup_{k \in \mathbb{N}} \{y_n, y_k^n\}$ is compact, we are done.

Since K is compact, it is bounded in Y. So by Proposition 1.4, $\overline{\operatorname{supp} K}$ is bounded in X. Since F is closed and discrete and X is normal, F is not bounded. This implies that there is $n \in \mathbb{N}$ such that $x_n \notin \overline{\operatorname{supp} K}$. Since X is regular there is $j_0 \in \mathbb{N}$ and a neighborhood V of $\overline{\operatorname{supp} K}$ such that $U_{j_0}^n \cap V = \varnothing$. So for every $z \in K$ and $j \geqq j_0$, f_j^n and the zero function on X are equal on V, which is a neighborhood of $\operatorname{supp}(z)$. Since ϕ is linear and effective, this implies that $g_j^n(z) = 0$ for every $j \geqq j_0$ and $z \in K$. But then $\sigma_n(y_k^n) = 0$ and $\sigma_n(y_n) = 0$, which gives a contradiction with $(*)$. □

By $X \oplus Y$ or $\oplus_{i=1}^{\infty} X_i$ we denote the topological sum of the topological spaces X and Y or $X_i (i \in \mathbb{N})$, respectively.

2.3 **Example.** In this example we show that the first countability condition in Theorem 2.2 is essential.

Let $X = \oplus_{i=1}^{\infty} [1, \omega]_i$. Let $A = X^{(1)}$ and $Y = Y_{X,A}$ the quotient space obtained from X by identifying A to single point, say ∞. Then X is clearly first countable and normal, and Y is normal but not first countable. By Lemma 1.1 we have $C_{p,A}(X) \sim C_{p,0}(Y)$. Furthermore we have

$$\begin{aligned} C_{p,A}(X) &\sim \prod_{i=1}^{\infty} C_{p,0}([1, \omega])_i \\ &\sim \prod_{i=1}^{\infty} C_p([1, \omega]) \quad \text{(Lemma 1.2a)} \\ &\sim C_p(X). \end{aligned}$$

Notice that for every Tychonov space Z and for every $z \in Z$, $C_p(Z) \sim C_{p,0}(Z) \times \mathbb{R}$, where $C_{p,0}(Z)$ consists of those functions in $C_p(Z)$ which vanish at z. So by Lemma 1.2, $C_p([1, \omega]) \sim C_p([1, \omega]) \times \mathbb{R}$. This implies $C_p(X) \sim C_p(X) \times \mathbb{R}$. So

$$\begin{aligned} C_p(X) &\sim C_p(X) \times \mathbb{R} \\ &\sim C_{p,A}(X) \times \mathbb{R} \\ &\sim C_{p,0}(Y) \times \mathbb{R} \\ &\sim C_p(Y). \end{aligned}$$

However $X^{(1)} = A$ is not countably compact, and $Y^{(1)} = \{\infty\}$ is countably compact.

From Theorem 2.2 and the result in [1] for normal spaces, that if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic and X is countably compact, then Y is countably compact, one could conjecture the following: Let α be an arbitary ordinal. If X and Y are both normal and first countable spaces such that

$C_p(X)$ and $C_p(Y)$ are linearly homeomorphic and $X^{(\alpha)}$ is countably compact, then $Y^{(\alpha)}$ is countably compact.

In the next example we show that if α is not a prime component, then the conjecture is false.

2.4 **Example.** Let $\alpha < \omega_1$ be an ordinal which is not a prime component. Observe that in this situation $1 \leqq \alpha' < \alpha$.

Let $X = \oplus_{i=1}^{\infty}[1, \omega^{\alpha'}]_i$ and $Y = \oplus_{i=1}^{\infty}[1, \omega^{\alpha}]_i$. By Theorem 1.3, $C_p[1, \omega^{\alpha'}] \sim C_p[1, \omega^{\alpha}]$, so that $C_p(X) \sim C_p(Y)$. But $Y^{(\alpha)} \approx \mathbb{N}$ (see [2] or [6] p. 155) which is not countably compact, and $X^{(\alpha)} = \varnothing$ which is countably compact.

Questions. (1) Is the above conjecture true for prime components?

(2) Does Theorem 2.2 still hold if normal is replaced by Tychonov?

References

1. A. V. Arhangelskiĭ, *On linear homeomorphisms of function spaces*, Soviet Math. Dokl. **25** (1982), 852–855.
2. J. Baars and J. de Groot, *An isomorphical classification of function spaces of zero dimensional locally compact separable metric spaces*, Comm. Math. Univ. Corolinae 29, 3 (1988), 577–595..
3. J. Baars, J. de Groot, J. van Mill and J. Pelant, *On topological and linear homeomorphisms of certain function spaces* (to appear in top. and appl.).
4. K. Kuratowski and A. Mostowski, *Set theory*, North-Holland Publishing Company, 1976.
5. Z. Semadeni, *Banach spaces of continuous functions*, PWN Warszawa, 1971.
6. W. Sierpiński, *Cardinal and ordinal numbers*, PWN Warszawa, 1958.

Faculteit Wiskunde en Informatica, Universiteit van Amsterdam, Roetersstraat 15, 1018 WB Amsterdam, The Netherlands

Faculteit Wiskunde en Informatica, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands