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AN INVARIANT OF DICHROMATIC LINKS

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ABSTRACT. We define a new polynomial invariant for a special class of dichromatic links. This polynomial generalizes the Jones polynomial.

A 1-*trivial dichromatic* link in S^3 is a link having at least two components, one of which is unknotted and labeled, or colored, "1", while all other components are colored "2". By using methods similar to those of Kauffman [K], we define a polynomial invariant of such links which is analogous to the Jones polynomial [J]. This polynomial has since been generalized by Hoste and Kidwell [H-K]. However, their approach is far more complicated, just as the establishment of the skein polynomial is more complicated than Kauffman's approach to the Jones polynomial [F-Y-H-L-M-O, P-T].

If L is a 1-trivial dichromatic link, then we may isotope L until the 1-*component*, that is the component colored "1", is the z-axis union the point at infinity. If we now project the link into the x-y plane we obtain a diagram of the 2-*sublink* in the punctured plane $\mathbf{R}^2 - \{0\}$. We may obviously use such *punctured* diagrams to represent 1-trivial dichromatic links. We may alter punctured diagrams by Reidemeister moves in $\mathbf{R}^2 - \{0\}$ and also by "flipping them over". That is, by replacing the projection of L with that of $\rho(L)$, where ρ is the 180° rotation of S^3 around the x-axis. The following theorem asserts that these alterations suffice to pass between all punctured diagrams of the same link.

Theorem 1. *Two punctured diagrams represent equivalent* 1-*trivial dichromatic links if and only if one can be transformed into the other by a finite sequence of Reidemeister moves in* $\mathbf{R}^2 - \{0\}$, *preceded by possibly flipping over one of the diagrams.*

Proof. Suppose D and D' represent 1-trivial dichromatic links L and L' whose 1-components L_1 and L_1' are the z-axis union the point at infinity and whose 2-sublinks L_2 and L_2' project to D and D', respectively. Let f be an

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orientation preserving homeomorphism of S^3 which takes L_1 to L_1' and L_2 to L_2'. Suppose first that f preserves the orientation of the z-axis. Clearly f is isotopic to a map f' which, in addition to taking L to L', is the identity on a regular neighborhood N of L_1. Let $X = \mathrm{cl}(S^3 - N)$ and consider f' restricted to X. This is a homeomorphism between solid tori which is the identity on the boundary. Since every homeomorphism between solid tori is, up to ambient isotopy, a power of a Dehn twist about a meridional disk, we see that f' is ambient isotopic to the identity. By modifying f' in a collar of X we may further assume that f' is ambient isotopic $\mathrm{rel}\,\partial X$ to the identity. This yields an isotopy of $S^3 \,\mathrm{rel}\, N$ which takes L_2 to L_2'. Projecting this isotopy into the x-y plane, we see D taken to D' in the complement of a disk centered at the origin. The usual proof that this can be accomplished by a finite sequence of Reidemeister moves can now be employed.

If originally f reverses the orientation of the z-axis then we may begin by flipping D' over. □

If D is a diagram, we denote by $\mathrm{sw}(D)$ the *self writhe* of D. This is the sum of the signs of those crossings between strands belonging to the same component.

Theorem 2. *There exists a unique polynomial invariant in* $\mathbf{Z}[A^{\pm 1}, h]$ *of unoriented* 1*-trivial dichromatic links given by*

$$d_L(A, h) = (-A^3)^{-\mathrm{sw}(D)} \langle D \rangle ,$$

where D *is any punctured diagram of the link* L *and* $\langle D \rangle$ *is the invariant of* D *determined by the following properties*:

1. $\langle \cdot \bigcirc \rangle = 1$,
2. $\langle \odot \rangle = h$,
3. $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$,
4. $\langle \cdot \bigcirc K \rangle = -(A^2 + A^{-2}) \langle \cdot K \rangle$, $K \neq \varnothing$,
5. $\langle \odot K \rangle = -(A^2 + A^{-2}) h \langle \cdot K \rangle$, $K \neq \varnothing$,

Here we follow Kauffman's notation [K] *with the additional convention of marking the puncture with a dot. Later, when working with ordinary diagrams, we will also subscript the components with their colors.*

Proof. Let D be a punctured diagram. Then, proceeding in a fashion similar to Kauffman [K], one sees that Properties 1–5 uniquely determine $\langle D \rangle$. Moreover, $\langle D \rangle$ is preserved by Type II and III Reidemeister moves as well as flipping over the diagram. However the effect of a Type I Reidemeister move is given by

$$\langle \succ\!\circ \rangle = -A^{-3} \langle \,) \,\rangle , \qquad \langle \succ\!\circ \rangle = -A^{3} \langle \,) \,\rangle .$$

From this it follows that d is a well-defined isotopy invariant of unoriented 1-trivial dichromatic links. □

Of course, d behaves similarly to the Jones polynomial with respect to connected sum, mutation, companionship, etc. Therefore, we list only a few additional properties of d.

1. Let L be a link represented by the punctured diagram D. Let $\overline{L}$ be represented by the diagram $\overline{D}$ obtained from D by reflecting in the plane of the projection. In other words, $\overline{D}$ is obtained from D by changing every crossing from over to under. Then

$$d_L(A, h) = d_{\overline{L}}(A^{-1}, h).$$

2. If one uses ordinary diagrams rather than punctured diagrams then the following "clasp" rule holds

$$A^2 d_{{}_i\times_j} + A^{-2} d_{{}_i\times_j} = (A^2 + A^{-2}) h d_{{}_i)(_j}, \qquad i \neq j.$$

3. Using ordinary (or punctured) diagrams, one has the following rule

$$(-A^3)^{\mathrm{sw}({}_2\times{}_2)} d_{{}_2\times_2} = A(-A^3)^{\mathrm{sw}({}_2)({}_2)} d_{{}_2)(_2} + A^{-1}(-A^3)^{\mathrm{sw}({}^2\asymp_2)} d_{{}_2\asymp_2}.$$

4. We may define an invariant $\tilde{d}$ of 1-trivial dichromatic links with oriented 2-sublink as follows. In general, if L is any link, some of whose components are oriented, let $\mathrm{lk}(L)$ denote the sum of the linking numbers between each pair of oriented components. Now let

$$\tilde{d}_L = (-A^3)^{-2\,\mathrm{lk}(L)} d_{|L|},$$

where $|L|$ denotes L stripped of its orientation. Then, again using ordinary (or punctured) diagrams, the following rule holds

$$A^4 \tilde{d}_{{}_2\times_2} - A^{-4} \tilde{d}_{{}_2\times_2} = (A^{-2} - A^2)\tilde{d}_{{}_2)(_2}.$$

5. If L is a 1-trivial dichromatic link, let $\mathrm{wrap}(L)$ be the wrapping number of the 2-sublink around the 1-component. That is, the minimal geometric intersection number of the 2-sublink with any disk spanning the 1-component. Then

$$\deg_h d_L \leq \mathrm{wrap}(L),$$

where $\deg_h$ is the highest degree of h appearing in d_L.

6. (a) If the 2-sublink L_2 is oriented then the Jones polynomial of L_2 is

$$V_{L_2}(A^{-1/4}) = \tilde{d}_L(A, 1).$$

(b) If L is oriented then the Jones polynomial of L is

$$V_L(A^{-1/4}) = -(A^2 + A^{-2})(-A^3)^{-2\,\mathrm{lk}(L)} d_{|L|}(A, (A^4 + A^{-4})/(A^2 + A^{-2})).$$

We mention two applications of d.

Suppose L is a link that is both 1-trivial and 2-trivial. In other words the 2-sublink is also an unknot. Hence we may compute d relative to either component. Call these two invariants d^1 and d^2, respectively. If L is interchangeable, that is, there is an isotopy exchanging the components, then $d^1_L = d^2_L$.

We may also use d to investigate periodic links.

Theorem 3. *Let r be prime. Suppose L is a 1-trivial dichromatic link invariant under a $\mathbf{Z}_r$-action on S^3 with fixed point set the 1-component. Then*

$$d_L(A, h) \equiv d_L(A^{-1}, h) \bmod (A^{4r} - 1, r).$$

In other words, the two polynomials differ by an element of the ideal generated by $A^{4r} - 1$ and r.

Proof. We can find an oriented punctured diagram D having r-fold rotational symmetry and such that $|D|$ represents L. Let $D_{\mathrm{sym}(\times)}$, $D_{\mathrm{sym}(\times)}$ and $D_{\mathrm{sym}()()}$ denote three punctured diagrams having r-fold rotational symmetry and which are identical except near the orbit of a single crossing where, at all r crossings, they appear instead with right, left and smoothed crossings respectively. Now using an idea of Murasugi's [M] (see also [P2]) and Property 4 we obtain

$$A^{4r}\,\tilde{d}_{D_{\mathrm{sym}(\times)}} - A^{-4r}\,\tilde{d}_{D_{\mathrm{sym}(\times)}} \equiv (A^{-2r} - A^{2r})\,\tilde{d}_{D_{\mathrm{sym}()()}} \bmod r.$$

Therefore

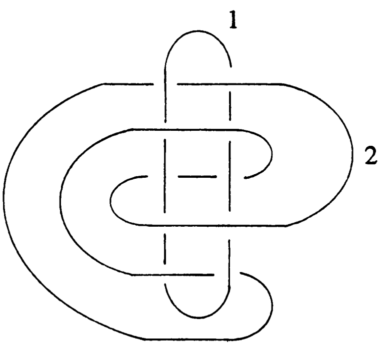
$$\tilde{d}_{D_{\mathrm{sym}(\times)}} \equiv \tilde{d}_{D_{\mathrm{sym}(\times)}} \bmod (A^{4r} - 1, r)$$

and hence

$$d_{D_{\mathrm{sym}(\times)}} \equiv d_{D_{\mathrm{sym}(\times)}} \bmod (A^{4r} - 1, r).$$

But this allows one to change $|D|$ to $|\overline{D}|$ without changing $d \bmod (A^{4r} - 1, r)$. Now applying Property 1 gives the desired result. □

Example. Let $L = 7_6^2$ with the components colored as shown below. Then $d^1 = -A^{12} + A^8 + A^{-4} + (A^{12} - A^8 + 1 - A^{-4})h^2$. It is laborious to compute d^2, but one can compute the coefficient of h^4 more easily. It equals $A^{16} + 2A^{12} - 2A^4 - 1$. Hence 7_6^2 is not interchangeable and the wrapping numbers are 2 and 4, respectively. By Theorem 3, there are no r-fold rotational symmetries about the 1-component with $r > 3$ or about the 2-component with $r > 2$ where r is prime.



The Link 7_6^2

Finally, we note that Theorem 2 can be interpreted in the language of skein modules [P1]. In particular, the theorem implies that the skein module,

$$\mathcal{S}_{2,\infty}(S^1 \times D^2, \mathbf{Z}[A^{\pm 1}])(A),$$

is a free module with infinite basis $\{h_i\}_{i=0}^{\infty}$, where h_0 is an unknot and h_i consists of i longitudes.

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