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ON q-DERANGEMENT NUMBERS

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ABSTRACT. We derive a q-analogue of the classical formula for the number of derangements of an n element set. Our derivation is entirely analogous to the classical derivation, but relies on a descent set preserving bijection between the set of permutations with a given derangement part and the set of shuffles of two permutations.

A classical application of binomial inversion (more generally the principle of inclusion–exclusion) is the derivation of the formula for the number of derangements (permutations with no fixed points) of an n element set:

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

This is obtained by counting permutations according to their number of fixed points and then inverting the resulting equation.

In this note we shall derive a formula of I. Gessel [G] for q-counting derangements by the major index statistic in a way entirely analogous to the classical $q = 1$ case. That is, we shall q-count permutations with k fixed points and then use Gauss inversion (q-binomial inversion or more generally Möbius inversion on the lattice of subspaces of a vector space) to derive the following formula for q-derangement numbers:

$$d_n(q) = [n]! \sum_{k=0}^{n} \frac{(-1)^k}{[k]!} q^{\binom{k}{2}}.$$

A key step in our derivation and an interesting result in its own right is a descent-preserving bijection between the set of permutations with a given derangement part and the set of shuffles of two permutations. This bijection enables us to use a formula of A. Garsia and I. Gessel for q-counting shuffles.

Gessel [G] obtained the formula for q-derangement numbers as a corollary of an Eulerian generating function formula for counting permutations by descents, major index, and cycle structure, which is proved via a correspondence

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between partitions and permutations. q-Derangement numbers have also been interpreted combinatorially on sets of permutations bijectively associated with derangements by A. Garsia and J. Remmel [GR] using the inversion index statistic and by J. Désarménien [D_2] (see [D_1] and [DW]) using the major index and inversion index statistics.

We shall briefly review some permutation statistic notation and terminology. For each integer $n \geq 1$, let $[n]$ denote the polynomial $1 + q + q^2 + \cdots + q^{n-1}$ and let $[n]!$ denote the polynomial $[n][n-1]\cdots[1]$. Also $[0]!$ is taken to be 1. The *q-binomial coefficients* are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

for $0 \leq k \leq n$.

For any positive integer n, let $\langle n \rangle$ denote the set $\{1, 2, \ldots, n\}$. We shall think of permutations in the symmetric group $\mathcal{S}_n$ as words with n distinct letters in $\langle n \rangle$. More generally, for a set A of n positive integers, $\mathcal{S}_A$ denotes the set of permutations of A or words with n distinct letters in A. The *descent set* of a permutation $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$ is $\operatorname{des}(\sigma) = \{i \in \langle n-1 \rangle | \sigma_i > \sigma_{i+1}\}$. The *major index* of σ is $\operatorname{maj}(\sigma) = \sum_{i \in \operatorname{des}(\sigma)} i$. Let us recall MacMahon's [M] formula for maj-q-counting permutations in $\mathcal{S}_n$:

$$\sum_{\sigma \in \mathcal{S}_n} q^{\operatorname{maj}(\sigma)} = [n]!.$$

A letter $i \in A$ is said to be a fixed point of $\sigma \in \mathcal{S}_A$ if $\sigma(i) = i$. A permutation with no fixed points is called a *derangement*. Let D_n denote the set of all derangements in $\mathcal{S}_n$. The *q-derangement* numbers are defined by

$$d_n(q) = \sum_{\sigma \in D_n} q^{\operatorname{maj}(\sigma)}.$$

It will be convenient to view the empty word Λ as a derangement and to define D_0 to be the set $\{\Lambda\}$. We also let $\operatorname{maj}(\Lambda) = 0$ and $d_0(q) = 1$.

For any permutation $\alpha \in \mathcal{S}_A$, where $A = \{a_1 < a_2 < \cdots < a_k\}$, define the *reduction* of α to be the permutation in $\mathcal{S}_k$ obtained from α by replacing each letter a_i by i, $i = 1, 2, \ldots, k$. The *derangement part* of a permutation $\sigma \in \mathcal{S}_n$, denoted $dp(\sigma)$, is the reduction of the subword of nonfixed points of σ. For example, $dp(5,3,1,4,7,6,2) =$ reduction of $5,3,1,7,2 = 4,3,1,5,2$. We shall use the convention that the derangement part of the identity permutation is the empty word Λ. Note that the derangement part of a permutation is a derangement, and that conversely, any derangement in D_k and k element subset of $\langle n \rangle$ determines a permutation in $\mathcal{S}_n$ with $n-k$ fixed points. Hence, the number of permutations in $\mathcal{S}_n$ with a given derangement part in D_k is $\binom{n}{k}$. Our goal is to q-count permutations with a given derangement part.

Let $\alpha \in D_k$. There is an obvious bijection between the set $\{\sigma \in \mathcal{S}_n | dp(\sigma) = \alpha\}$ and the set $\operatorname{Sh}(\alpha, \beta)$ of all *shuffles* of α and $\beta = k+1, k+2, \ldots, n$, i.e. permutations in $\mathscr{S}_n$ which contain α and β as complementary subwords. Indeed, for each permutation σ in the former set, replace the subword of non-fixed points of σ by α and the complementary subword of fixed points by β. A very useful result of Garsia and Gessel [GG, Theorem 3.1] allows us to q-count the latter set. Unfortunately, since the above-mentioned bijection does not preserve the major index, it does not help us in q-counting the former set. However, we shall show that there is another bijection between these two sets of permutations which, in fact, preserves descent sets.

Define a letter σ_i of $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n \in \mathscr{S}_n$ to be an *excedant* of σ if $\sigma_i > i$ and a *subcedant* of σ if $\sigma_i < i$. Let $s(\sigma)$ and $e(\sigma)$ be the number of subcedants and excedants, respectively, of σ. We now fix n and let $k \le n$. For $\sigma \in \mathscr{S}_k$, let $\tilde{\sigma}$ be the permutation of k letters obtained from σ by replacing its ith smallest subcedant by i, $i = 1, 2, \ldots, s(\sigma)$, its ith smallest fixed point by $s(\sigma) + i$, $i = 1, 2, \ldots, k - s(\sigma) - e(\sigma)$, and its ith largest excedant by $n - i + 1$, $i = 1, 2, \ldots, e(\sigma)$. Note that $\tilde{\sigma}$ depends on n as well as σ. For example, if $\sigma = \overline{3}2\overline{65}\underline{41}$ (with subcedants underlined and excedants overlined) and $n = 8$ then $\tilde{\sigma} = 638721$. If $k = n$ then $\tilde{\sigma} \in \mathscr{S}_n$. If σ is a derangement then $\tilde{\sigma} \in \mathscr{S}_A$, where $A = \{1, 2, \ldots, s(\sigma)\} \cup \{n - e(\sigma) + 1, n - e(\sigma) + 2, \ldots, n\}$.

Lemma 1. *Let* $\sigma \in \mathscr{S}_k$, $k \le n$. *Then* $\operatorname{des}(\sigma) = \operatorname{des}(\tilde{\sigma})$.

Proof. Suppose $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_k$ and $\tilde{\sigma} = \tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_k$. For each $i \in \langle k - 1 \rangle$, we shall show $i \in \operatorname{des}(\sigma)$ if and only if $i \in \operatorname{des}(\tilde{\sigma})$, by considering the nine possible designations of subcedant (s), excedant (e), and fixed point (f) to σ_i and σ_{i+1}. First note that if σ_i is a subcedant of σ then $\tilde{\sigma}_i \le \sigma_i$ and if σ_i is an excedant of σ then $\tilde{\sigma}_i \ge \sigma_i$.

Cases 1–3. Suppose (σ_i, σ_{i+1}) is an (s, s), (e, e), or (f, f) pair. It is then clear that $\sigma_i < \sigma_{i+1}$ if and only if $\tilde{\sigma}_i < \tilde{\sigma}_{i+1}$.

Case 4. Suppose (σ_i, σ_{i+1}) is a (s, e) pair. Then we have

$$\tilde{\sigma}_i \le \sigma_i < i < i + 1 < \sigma_{i+1} < \tilde{\sigma}_{i+1},$$

which shows that $i \notin \operatorname{des}(\sigma)$ and $i \notin \operatorname{des}(\tilde{\sigma})$.

Case 5. Suppose (σ_i, σ_{i+1}) is a (s, f) pair. Now we have

$$\sigma_i < i < i + 1 = \sigma_{i+1} \quad \text{and} \quad \tilde{\sigma}_i \le s(\sigma) < \tilde{\sigma}_{i+1},$$

which shows that $i \notin \operatorname{des}(\sigma)$ and $i \notin \operatorname{des}(\tilde{\sigma})$.

Case 6. Suppose (σ_i, σ_{i+1}) is a (f, s) pair. Then since $\sigma_{i+1} < i+1$ and $\sigma_i = i$, we have

$$\sigma_{i+1} < \sigma_i \quad \text{and} \quad \tilde{\sigma}_{i+1} \le s(\sigma) < \tilde{\sigma}_i.$$

This shows that $i \in \operatorname{des}(\sigma)$ and $i \in \operatorname{des}(\tilde{\sigma})$.

Cases 7–9. The remaining three cases are that (σ_i, σ_{i+1}) is a (f, e), (e, s), or (e, f) pair. These cases are handled similarly to the previous three cases and are left to the reader. □

Theorem 2. *Let* $\alpha \in D_k$, $k \leq n$, *and* $\gamma = s(\alpha)+1, s(\alpha)+2, \dots, n-e(\alpha)$. *Then the map* $\varphi\colon \{\sigma \in \mathscr{S}_n | dp(\sigma) = \alpha\} \to \mathrm{Sh}(\tilde{\alpha}, \gamma)$ *defined by* $\varphi(\sigma) = \tilde{\sigma}$ *is a bijection which preserves descent sets, i.e.* $\mathrm{des}(\sigma) = \mathrm{des}(\varphi(\sigma))$. *Consequently, for all* $J \subseteq \langle n-1 \rangle$,

$$|\{\sigma \in \mathscr{S}_n | dp(\sigma) = \alpha,\ \mathrm{des}(\sigma) = J\}| = |\{\sigma \in \mathrm{Sh}(\tilde{\alpha}, \gamma) | \mathrm{des}(\sigma) = J\}|.$$

Proof. In view of Lemma 1, we need only show that φ is an invertible map whose image is $\mathrm{Sh}(\tilde{\alpha}, \gamma)$. First, we claim that if $dp(\sigma) = \alpha$ then $\tilde{\sigma}$ is obtained from σ by replacing the subword of nonfixed points of σ by $\tilde{\alpha}$ and the subword of fixed points of σ by γ. Indeed, the subword of fixed points of σ is replaced by the word $s(\sigma)+1, s(\sigma)+2, \dots, n-e(\sigma)$, which is precisely γ since $s(\sigma) = s(\alpha)$ and $e(\sigma) = e(\alpha)$. Also since α is the reduction of the subword of nonfixed points of σ, the position of the ith smallest subcedant of α is the same as the position of the ith smallest subcedant of σ in the subword of nonfixed points. The same is true for the ith smallest excedant. Hence each subcedant and excedant of σ is replaced by the same letter that replaces the corresponding subcedant or excedant of α. This means that the subword of subcedants and excedants of σ is replaced by $\tilde{\alpha}$. We may now conclude that $\tilde{\sigma} \in \mathrm{Sh}(\tilde{\alpha}, \gamma)$.

The above description of $\tilde{\sigma}$ as a shuffle of $\tilde{\alpha}$ and γ also implies that φ is invertible. Indeed, if we replace the $\tilde{\alpha}$ subword of any $\tau \in \mathrm{Sh}(\tilde{\alpha}, \gamma)$ by the permutation, of the subword positions, whose reduction is α, and the letters of the γ subword by their positions, we obtain a unique permutation $\sigma \in \mathscr{S}_n$ such that $dp(\sigma) = \alpha$ and $\varphi(\sigma) = \tau$. □

Remark. Although a descent set preserving bijection between $\{\sigma \in \mathscr{S}_n | dp(\sigma) = \alpha\}$ and $\mathrm{Sh}(\alpha, \beta)$, where $\beta = k+1, k+2, \dots, n$, will not be needed in the sequel, we should point out here that one can be constructed by composing the bijection φ with a bijection between $\mathrm{Sh}(\alpha, \beta)$ and $\mathrm{Sh}(\tilde{\alpha}, \gamma)$ described in [BW, Proof of Proposition 4.1].

Corollary 3. *Let* $\alpha \in D_k$ *and* $k \leq n$. *Then*

$$\sum_{\substack{dp(\sigma)=\alpha \\ \sigma \in \mathscr{S}_n}} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}(\alpha)} \begin{bmatrix} n \\ k \end{bmatrix}.$$

Proof. Since $\mathrm{maj}(\sigma)$ depends only on $\mathrm{des}(\sigma)$, it follows from Theorem 2 that

$$\sum_{dp(\sigma)=\alpha} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in \mathrm{Sh}(\tilde{\alpha}, \gamma)} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}(\tilde{\alpha})} \begin{bmatrix} n \\ k \end{bmatrix},$$

with the last step following from Garsia–Gessel [GG, Theorem 3.1]. (For a bijective alternative proof and generalization of the Garsia–Gessel result, see [BW].) By Lemma 1, $\mathrm{maj}(\tilde{\alpha}) = \mathrm{maj}(\alpha)$, which completes the proof. □

Theorem 4. *For all* $n \geq 0$,

$$d_n(q) = [n]! \sum_{k=0}^{n} \frac{(-1)^k}{[k]!} q^{\binom{k}{2}}.$$

Proof. By maj-q-counting the permutations in $\mathscr{S}_n$ according to derangement part and applying Corollary 3, we obtain

$$\begin{aligned}
[n]! &= \sum_{\sigma\in\mathscr{S}_n} q^{\operatorname{maj}(\sigma)} \\
&= \sum_{k=0}^{n} \sum_{\alpha\in D_k} \sum_{dp(\sigma)=\alpha} q^{\operatorname{maj}(\sigma)} \\
&= \sum_{k=0}^{n} \sum_{\alpha\in D_k} q^{\operatorname{maj}(\alpha)} \begin{bmatrix} n \\ k \end{bmatrix} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} d_k(q).
\end{aligned}$$

Gauss inversion [A, p. 96] on the resulting equation yields,

$$\begin{aligned}
d_n(q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} q^{\binom{n-k}{2}} [k]! \\
&= \sum_{k=0}^{n} \frac{[n]!}{[n-k]!} (-1)^{n-k} q^{\binom{n-k}{2}},
\end{aligned}$$

which is equivalent to the desired formula. □

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References

[A] M. Aigner, *Combinatorial theory*, Springer-Verlag, New York, 1979.

[BW] A. Björner and M. Wachs, *q-Hook length formulas for forests*, J. Combin. Th. Ser A (to appear).

[D$_1$] J. Désarménien, Une autre interprétation du nombre de dérangements, *Actes* 8^e *Séminaire Lotharingien de Combinatoire*, I.R.M.A. Strasbourg, 1984, pp. 11–16.

[D$_2$] J. Désarménien, personal communication.

[G] I. Gessel, *Counting permutations by descents, greater index, and cycle structure*, unpublished work.

[GG] A. M. Garsia and I. Gessel, *Permutation statistics and partitions*, Adv. in Math. **31** (1979), 288–305.

[GR] A. M. Garsia and J. Remmel, *A combinatorial interpretation of q-derangement and q-Laguerre numbers*, European J. Combin. **1** (198C), 47–59.

[M] P. A. MacMahon, *The indices of permutations and the derivation therefrom of functions of a single variable associated with permutations of any assemblage of objects*, Amer. J. Math. **35** (1913), 281–322; reprinted in *Percy Alexander MacMahon: Collected papers*, vol. 1 (G. E. Andrews, ed.), M.I.T. Press, Cambridge MA., 1978, pp. 508–549.

[DW] J. Désarménien and M. Wachs, *Descentes des dérangements et mots circulaires*, Actes 19 Séminaire Lotharinjien de Combinatoire, I.R.M.A. Strasbourg, 1988, 13–21.

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