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A GENERALIZATION OF THE WEDDERBURN-ARTIN THEOREM

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ABSTRACT. The structure of rings such that each of its homomorphic images has the property that each cyclic right module over it is essentially embeddable in a direct summand is determined. Such rings are precisely (i) right uniserial rings, (ii) $n \times n$ matrix rings over two-sided uniserial rings with $n > 1$, or (iii) sums of rings of the types (i) and (ii).

1. INTRODUCTION

In this paper we study rings R with the following property (P): For all homomorphic images $\overline{R}$ of R, every cyclic right $\overline{R}$-module is essentially embeddable in a direct summand of $\overline{R}$. Our results generalize the celebrated Wedderburn-Artin theorem which characterizes rings R such that over all the homomorphic images $\overline{R}$ the cyclic modules are isomorphic to direct summands of $\overline{R}$. Examples of rings satisfying (P) include semisimple artinian rings and right uniserial rings. Indeed we show that a ring R has property (P) if and only if R is a direct sum of right uniserial rings and matrix rings over right self-injective right uniserial rings if and only if R is a semiperfect ring whose cyclic right modules are essentially embeddable in direct summands (Theorem 3.5). Throughout this paper, all rings have 1 and all modules are right unital, unless otherwise stated. By a right (left) uniserial ring, we mean a ring having a unique composition series of right (left) ideals. A ring which is both right and left uniserial will simply be called uniserial. A right uniserial ring is uniserial iff it is right self-injective. For any module $M, E(M), \mathrm{Soc}(M)$ and $J(M)$ will denote, respectively, the injective hull, the socle, and the Jacobson radical of M.

2. PRELIMINARY RESULTS

Throughout this section, we assume that R is a ring satisfying property (P).

2.1. **Lemma.** *R is a semiperfect ring.*

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Proof. Let $N =$ prime radical of R under our hypothesis, each right ideal of R/N is an annihilator right ideal and hence R is semiperfect [3, p. 204, Exercise 24.3(d)–(e)]. $\square$

Since R is semiperfect, R has a complete orthogonal set $e_1, \ldots, e_n$ of idempotents such that, for all i, e_iRe_i is a local ring. In the lemmas which follow the decomposition $R = e_1R \oplus \cdots \oplus e_nR$ will be frequently used. For R modules A and B, the notation $A \hookrightarrow' B$ shall mean that A is essentially embeddable in B.

2.2. **Lemma.** *For $R = e_1R \oplus \cdots \oplus e_nR$, the following are true:*

(i) *e_iR is uniform for all i,*
(ii) *$\operatorname{Soc} R$ is essential in R, and*
(iii) *R has Goldie dimension n.*

Proof. Let $\mathbf{S} = \{S_1, \ldots, S_k\}$ be an irreduntant set of representatives for the simple R-modules and let $\mathbf{P} = \{e_1R, \ldots, e_kR\}$ be a complete set of representatives for the projective indecomposable R modules.

Since every simple module S is cyclic, it is essentially embeddable in eR for some idempotent $e \in R$. Clearly eR is indecomposable. Thus we can define a function $f\colon \mathbf{S} \to \mathbf{P}$ by $f(S_i) = e_jR$ where $S_i \hookrightarrow' e_jR$. The function f must be one to one, hence onto. It easily follows that each e_jR $(j = 1, \ldots, n)$ contains an essential simple submodule T_j and, therefore, each e_jR is uniform. Also, $T_1 \oplus \cdots \oplus T_n = \operatorname{Soc} R$ is essential in R. Thus R has Goldie dimension n. $\square$

2.3. **Lemma.** *R is right artinian.*

Proof. Clearly each cyclic R-module has nonzero socle. Thus, R is left perfect because R is semiperfect [2]. Furthermore, since $J(R)/(J(R))^2$ is completely reducible, $J(R)/(J(R))^2$ is embeddable in $\operatorname{Soc} R$. This yields $J(R)/(J(R))^2$ is finitely generated and so R is right artinian [1, p. 322]. $\square$

2.4. **Lemma.** *For $i \neq j$, let e_iR and e_jR be indecomposable summands of R. Then, either e_iR is isomorphic to e_jR or $\operatorname{Hom}_R(e_iR, e_jR) = 0$.*

Proof. Suppose $\sigma\colon e_iR \to e_jR$ is not zero, then $e_iR/\operatorname{Ker}\sigma$ is embeddable in e_jR. Since e_jR is uniform (Lemma 2.2), such an embedding must be essential. This implies $E(e_iR/\operatorname{Ker}\sigma) \cong E(e_jR)$. Also, since R satisfies property (P) and it has Goldie dimension n, $E(R/\operatorname{Ker}\sigma) \cong E(R)$. Let $R = e_1R \oplus \cdots \oplus e_nR$. Then

$$R/\operatorname{Ker}\sigma \cong e_1R \oplus \cdots \oplus e_iR/\operatorname{Ker}\sigma \oplus \cdots \oplus e_jR \oplus \cdots \oplus e_nR,$$

which yields

$$\begin{aligned} (1)\qquad & E(e_1R) \oplus \cdots \oplus E(e_jR) \oplus \cdots \oplus E(e_jR) \oplus \cdots \oplus E(e_nR) \\ \cong E(R/\operatorname{Ker}\sigma) \cong E(R) \cong {} & E(e_1R) \oplus \cdots \oplus E(e_iR) \oplus \cdots \oplus E(e_jR) \oplus \cdots \oplus E(e_nR). \end{aligned}$$

Since e_kR is uniform for all k, $E(e_kR)$ has local endomorphism ring. Hence from (1) $E(e_iR) \cong E(e_jR)$. But this implies that $E(e_iR)$ and $E(e_jR)$ contain isomorphic copies of the same simple submodule S and, therefore, e_iR and e_jR both contain essentially a copy of S. This implies that e_iR is isomorphic to e_jR. □

2.5. **Lemma.** *R is a direct sum of matrix rings over local rings.*

Proof. Let $[e_iR] = \sum e_jR$, where the $\sum$ runs over all j for which $e_jR \cong e_iR$. Renumbering if necessary we may write

$$R = [e_1R] \oplus \cdots \oplus [e_kR]$$

where $k \leq n$. By Lemma 2.4, $[e_1R]$ is an ideal in R and so

$$R \cong M_{n_1}(e_1Re_1) \oplus \cdots \oplus M_{n_k}(e_kRe_k)$$

where n_i is the number of summands in $[e_iR]$. □

Next we proceed to show that each local ring e_iRe_i is indeed right uniserial.

2.6. **Lemma.** *If $R = S_n$ is the $n \times n$ matrix ring over a local ring S, then S is right uniserial.*

Proof. Write $R = e_{11}R \oplus \cdots \oplus e_{nn}R$, where $e_{11}, e_{22}, \ldots, e_{nn}$ are the usual matrix units. Notice that each $e_{ii}R$ is indecomposable since S is local.

Consider $I \subset e_{11}R$. Then $R/I \cong e_{11}R/I \times e_{22}R \times \cdots \times e_{nn}R$ is essentially embeddable in R because the Goldie dimension of R is n. Thus

$$E(R/I) \cong E(R)$$

and so

$$E(e_{11}R/I) \times E(e_{22}R) \times \cdots \times E(e_{nn}R) \cong E(e_{11}R) \times E(e_{22}R) \times \cdots \times E(e_{nn}R).$$

Since $e_{ii}R$ is uniform (Lemma 2.2), $E(e_{ii}R)$ is also uniform. Therefore, by Azumaya diagram, $E(e_{11}R/I) \cong E(e_{11}R)$. This implies $e_{11}R/I$ is uniform. It follows that the submodules of $e_{11}R$ are linearly ordered. We show now that $S \cong e_{11}Re_{11}$ is right uniserial. Let A, B be right ideals of $e_{11}Re_{11}$. Then $Ae_{11}R \subset e_{11}R$ and $Be_{11}R \subset e_{11}R$ and so either $Ae_{11}R \subset Be_{11}R$ or $Be_{11}R \subset Ae_{11}R$. But then either $A = Ae_{11}Re_{11} \subset Be_{11}Re_{11} = B$ or $B = Be_{11}Re_{11} \subset Ae_{11}Re_{11} = A$, proving our assertion. □

In the next section we shall obtain a characterization of rings with property (P).

2.7. *Remark.* Note that in the proof of Lemmas 2.2–2.6 we have only used that R is a semiperfect ring each of whose cyclic R-modules is essentially embeddable in a direct summand of R.

3. Main results

We begin with

3.1. Theorem. *Let R be a ring with property* (P). *Then R is a direct sum of matrix rings over right uniserial rings.*

Proof. The proof follows from Lemmas 2.5, 2.6, 2.7 and the fact that ring direct summands of a ring with property (P) inherit the property (P). □

It is obvious that right uniserial rings have property (P). In what follows we will concentrate on showing that for a right uniserial ring S, the matrix ring $R = S_n$ $(n > 1)$ satisfies property (P) if and only if S is right self-injective. For the sake of our discussion we define property (Q) for modules. We say that an R-module M has property (Q) if each factor of M is essentially embeddable in a direct summand of M.

3.2. Lemma. *The $n \times n$ matrix ring over R has property* (Q) *as a module over itself if and only if the R-module $R^{(n)}$ has property* (Q).

Proof. Given a category isomorphism $F = \mathscr{M}_S \to \mathscr{M}_T$ between the categories of right modules of two rings S and T, it is obvious that a module $M \in \mathscr{M}_S$ satisfies (Q) if and only if $F(M) \in \mathscr{M}_T$ satisfies (Q). Our lemma follows from the fact that if $e_{11} \in R_n$ is the usual matrix unit then $R^{(n)} \in \mathscr{M}_R$ corresponds to $R_n \in \mathscr{M}_{R_n}$ under the category isomorphism.

$$- \otimes_{R_n} R_n e_{11} : \mathscr{M}_{R_n} \to \mathscr{M}_R . \quad \square$$

3.3. Lemma. *If the R-module $R^{(n)}$ has property* (Q) *where R is right uniserial and $n > 1$, then R is right self-injective.*

Proof. Let R be a right uniserial ring which is not right self-injective. Then there exists $s \in R$ such that $xs \notin Rx$. Without loss of generality, we may assume that s is invertible. Define $I = (x, -xs, 0, 0, \dots, 0)R \subseteq R^{(n)}$. We claim that $R^{(n)}/I$ is not embeddable in $R^{(n)}$. Notice that both $\bar{e}_1 R$ and $\bar{e}_2 R$ are isomorphic to R as R-modules, where $e_1 = (1, 0, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$. Also, since $\bar{e}_1 R \cap \bar{e}_2 R = \bar{e}_1 x R_1 = \bar{e}_2 x R$. If $\psi: R^{(n)}/I \to R^{(n)}$ were an embedding of $R^{(n)}/I$ into $R^{(n)}$, and if $\psi(\bar{e}_1) = (a_1, a_2, \dots, a_n)$ and $\psi(\bar{e}_2) = (b_1, b_2, \dots, b_n)$, then there must exist i, j such that a_i invertible and b_j invertible. However, $\psi(\bar{e}_1 x) = (a_1 x, a_2 x, \dots, a_n x)$ and $\psi(\bar{e}_2 xs) = (b_1 xs, b_2 xs, \dots, b_n xs)$, which implies that $a_j x = b_j xs$. Hence $b_j^{-1} a_j x = xs$, contradicting our choice of s. So we have shown that the R-module $R^{(n)}$ does not satisfy (Q). □

3.4. Lemma. *If R is a right self-injective right uniserial ring, then R_n satisfies property* (P).

Proof. Since R is self-injective, it follows that R_n is also self-injective. Therefore, R_n satisfies property (Q) as a module over itself if and only if the injective hull of any cyclic R_n-module is embeddable in R_n. Let $e_{11} \in R_n$ be the usual matrix unit and let I be a right ideal of R_n. Since $R_n \to R_n/I \to 0$ is exact, $(R_n \otimes_{R_n} R_n e_{11})_R \to (R_n/I \otimes_{R_n} R_n e_{11})_R \to 0$ is also exact. But $(R_n \otimes_{K_n} R_n e_{11})_R \cong (R_n e_{11})_R \cong R^{(n)}$. Therefore, $N = R_n/I \otimes_{R_n} R_n e_{11}$ is a homomorphic image of $R^{(n)}$. Thus N is an extension of a sum of k cyclic R-modules, $(k \leq n)$ [5, Lemma 1.16]. But then, since $e_{11}R_n$ corresponds to R under $\mathrm{Hom}_R(R_n e_{11}, \_)$, the inverse of $(\_ \otimes_{R_n} R_n e_{11})$, it follows that there exist k quotients $Q_1, \ldots, Q_k$, of $e_{11}R_n$ such that $Q_1 \oplus \cdots \oplus Q_k \hookrightarrow' R_n/I$. Now, $E(Q_i) \hookrightarrow' e_{11}R_n$ for all i. Hence $E(R_n/I) = E(Q_1) \oplus \cdots \oplus E(Q_k) \hookrightarrow' (e_{11}R_n)^{(k)} \hookrightarrow R_n$, proving that $E(R_n/I)$ is embeddable in R_n. Since each homomorphic image of R is again right self-injective right uniserial, it follows that R_n satisfies property (P). □

Our results are summarized in the following theorem.

3.5. **Theorem.** *A ring R satisfies* (P) *if and only if R is a direct sum of right uniserial rings and matrix rings over right self-injective right uniserial rings if and only if R is a semiperfect ring whose cyclics are essentially embeddable in a direct summand of R.*

Proof. The proof follows from Theorem 3.1 and Lemmas 3.2, 3.3 and 3.4 and Remark 2.7. □

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