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DEGREES OF IRREDUCIBLE CHARACTERS AND NORMAL p-COMPLEMENTS

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ABSTRACT. John Tate [1] proved that if $P \in \mathrm{Syl}_p(G)$, H is a normal subgroup of a finite group G and $P \cap H \le \Phi(P)$ ($\Phi(G)$ is the Frattini subgroup of G) then H has a normal p-complement. We prove in this note that Tate's theorem has nice character-theoretic applications.

Theorem. *Let B be the intersection of the kernels of all nonlinear irreducible characters of G with p'-degree. Then $B \cap G' \cap P \subseteq P'$ where $P \in \mathrm{Syl}_p(G)$. Also, B has a normal p-complement.*

Proof. We suppose that $P_0 = B \cap G' \cap P \nleq P'$. Let $\mathrm{Lin}(P)$ be the set of all linear characters of P, and let $\lambda \in \mathrm{Lin}(P)$ satisfy $P_0 \nleq \ker \lambda$. Then the induced character λ^G has degree $|G\colon P| \not\equiv 0 \pmod p$. Let χ be an irreducible component of λ^G. Then $P_0 \nleq \ker \chi$ by Frobenius reciprocity. So p divides $\chi(1)$ for all nonlinear irreducible components χ of λ^G. Since p does not divide $\lambda^G(1)$, the character λ^G has a linear component λ^0. Then $\lambda^0_P = \lambda$. Thus

$$P \cap \ker \lambda^0 = \ker \lambda \ngeq P_0 \Rightarrow P_0 \nleq \ker \lambda^0.$$

Since $G' \le \ker \lambda^0$, we have

$$B \cap G' \cap P = P_0 \nleq G',$$

which is a contradiction.

The last assertion follows from

Lemma. *Let $P \in \mathrm{Syl}_p(G)$ and let $H \trianglelefteq G$. If $H \cap G' \cap P \le P'$, then H has a normal p-complement.*

Proof. Let $O^p(G)$ be the intersection of all $N \trianglelefteq G$ such that G/N is a p-group. Then $O^p(G)$ is characteristic in G. So $O^p(H) \trianglelefteq G$ and

$$O^p(H) \cap G' \cap P \le H \cap G' \cap P \le P'.$$

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Since $O^p(H)$ has no normal subgroup of index p, we have $O^p(H) \cap P \leq G'$. Hence

$$O^p(H) \cap G' \cap P = O^p(H) \cap P$$

and H has a normal p-complement by Tate's theorem.

Corollary (J. G. Thompson [2]). *Suppose that a prime p divides $\chi(1)$ for all nonlinear irreducible characters χ of G. Then G has a normal p-complement.*

This follows from Theorem, since $B = G$, where B is defined in the theorem.

Remark. We prove that Tate's theorem for $p > 2$ is a corollary to the following well-known result of J. G. Thompson [3]:

Let $p > 2$, let $P \in \mathrm{Syl}_p(G)$, and, for every characteristic subgroup P_0 of P, $P_0 \neq 1$, the normalizer $N_G(P_0)$ has a normal p-complement. Then G has a normal p-complement.

Suppose that $H \trianglelefteq G$, $p > 2$, $P \in \mathrm{Syl}_p(G)$, and $P_1 = H \cap P \leq \Phi(P)$. Suppose that H has no normal p-complement. By Thompson's theorem, there exists a characteristic subgroup P_0 of P_1, $P_0 \neq 1$, such that $N_H(P_0)$ has no normal p-complement, and let P_0 have a maximal order among all subgroups with this property. Since $P_1 \trianglelefteq P$, we have $P_0 \trianglelefteq P$. So $P < N_G(P_0)$. Since $N_H(P_0) \leq N_G(P_0)$, the subgroup $N_G(P_0)$ has no normal p-complement. Without loss of generality we may assume that $PH = G$. Then

$$N_G(P_0) = P(H \cap N_G(P_0)) = PN_H(P_0) = N_H(P_0)P$$

by modular law. Since $N_G(P_0)$ has no normal p-complement we may assume without loss that $N_G(P_0) = G$. So $P_0 \trianglelefteq G$. Suppose that $P_0 \not\leq \Phi(G)$. Then there exists such a maximal subgroup M of G that $P_0 M = G$. Then $P = P_0(P \cap M)$ by modular law. So $P \cap M = P$ (since $P_0 \leq \Phi(P)$), and $P_0 \leq M$, $M = P_0 M = G$, a contradiction. Hence $P_0 \leq \Phi(G)$. By Thompson's theorem G/P_0 has a normal p-complement T/P_0 by virtue of a maximal choice of P_0. If K is a p'-Hall subgroup of T (Schur–Zassenhaus), then

$$G = N_G(K)T = N_G(K)KP_0 = N_G(K)P_0$$

(Schur–Zassenhaus and Frattini). Since $P_0 \leq \Phi(G)$, we have $N_G(K) = G$ and $K \trianglelefteq G$. Obviously, K is a normal p-complement of G.

Further applications of a generalization of Tate's theorem (Roquette's theorem [4]) can be found in Chapter 6 of the book [5].

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