

THE NORMAL INDEX OF A MAXIMAL SUBGROUP OF A FINITE GROUP

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ABSTRACT. For a maximal subgroup M of a finite group G , the *normal index* $\eta(G : M)$ is defined to be the order of a chief factor H/K where H is minimal in the set of supplements of M in G . We obtain several results on the normal index of maximal subgroups M of composite index in G with $[G : M]_p = 1$ which imply G to be solvable, supersolvable.

1. INTRODUCTION AND NOTATION

If M is a maximal subgroup of a finite group G , then the normal index $\eta(G : M)$ is defined in Deskins [6] to be the order of a chief factor H/K where H is minimal in the set of supplements of M in G . It was shown by Deskins [6, 2.5] that a group G is solvable if and only if $\eta(G : M) = [G : M]$ for each maximal subgroup M of G . Now, it is interesting to investigate whether G is solvable if and only if the hypothesis $\eta(G : M) = [G : M]$ is satisfied by only a certain subclass of maximal subgroups of G . This was proved in [12, Theorem 2.3] for the subfamily of maximal subgroups M of G such that $[G : M]$ is composite. Here, we extend this and some other results about the normal index from [2, 8 and 12] to the case when M is a maximal subgroup of composite index such that $[G : M]_p = 1$ where p is a given prime. Such families of maximal subgroups have been considered by us in [4, 9–11] in connection with developing an analog of the *Frattini Subgroup* of a finite group.

If $M \leq G$ we denote $M < \cdot G$ to indicate that M is a maximal subgroup of G . For a group G and any prime p , we shall consistently use the following

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notation for the three families of maximal subgroups defined below.

$$\begin{aligned}\Lambda(G) &= \{M : M < \cdot G : [G : M] \text{ is composite}\} \\ \Upsilon_p(G) &= \{M : M < \cdot G : [G : M]_p = 1\} \\ \Xi_p(G) &= \Lambda(G) \cap \Upsilon(G)\end{aligned}$$

2. PRELIMINARIES

If M is a maximal subgroup of a group G and H is a minimal supplement to M then for any chief factor H/K of G it follows that $K \subseteq M$ and $G = MH$. Therefore $[G : M]$ divides $o(H/K) = \eta(G : M)$. If G is simple then obviously $\eta(G : M) = o(G)$ for any maximal subgroup M of G . For any group G the integer $\eta(G : M)$ is uniquely determined by the maximal subgroup M [6, 2.1].

Lemma 2.1 [2, Lemma 2]. *If $N \triangleleft G$ and M is a maximal subgroup of G such that $N \subseteq M$, then*

$$\eta(G/N : M/N) = \eta(G : M).$$

Lemma 2.2 [12, Lemma 3.1]. *If M is a maximal subgroup of a group G such that $[G : M]$ is a square-free integer then*

$$\eta(G : M) = [G : M].$$

Definition. Let G be any group and p be any prime. Define three characteristic subgroups of G as follows:

$$\begin{aligned}S_p(G) &= \bigcap \{M : M \in \Xi_p(G)\}, \\ L(G) &= \bigcap \{M : M \in \Lambda(G)\}, \\ \Phi_p(G) &= \bigcap \{M : M \in \Upsilon_p(G)\}.\end{aligned}$$

In case $\Xi_p(G)$ is empty then we define $S_p(G) = G$ and the same thing is done for the other two subgroups.

The subgroup $\Phi_p(G)$ was introduced in [6], $L(G)$ was defined in [3] and $S_p(G)$ has been developed in [4, 9–11]. All these subgroups contain the Frattini subgroup $\Phi(G)$ and further $S_p(G)$ contains both $L(G)$ and $\Phi_p(G)$.

Lemma 2.3 [9, Corollary 4]. *Let $K \triangleleft G$. If $K \subseteq S_p(G)$ then*

$$S_p(G/K) = S_p(G)/K.$$

Theorem 2.4 [9, Theorem 8]. (i) *If p is the largest prime dividing the order of a group G then $S_p(G)$ is solvable.* (ii) *If G is p -solvable, then $S_p(G)$ is solvable.*

Proposition 2.5 [9, Proposition 5]. *If q is the largest prime dividing the order of $S_p(G)$ where p is an arbitrary prime, then any Sylow q -subgroup of $S_p(G)$ is normal in G .*

Recall that a group G is called a Sylow tower group of supersolvable type if the following conditions hold.

- (i) $p_1 > p_2 > \dots > p_r$ are all the distinct prime divisors of the order of G and P_i is a Sylow p_i -subgroup of G , $1 \leq i \leq r$.
- (ii) $P_1 P_2 \dots P_k \triangleleft G$, $k = 1, 2, \dots, r$.

Proposition 2.6. *If p is the largest prime dividing the order of a group G , then $S_p(G)$ is a Sylow tower group of supersolvable type.*

Proof. We distinguish two cases.

Case 1. p divides the order of $S_p(G)$. Let P be a Sylow p -subgroup of $S_p(G)$. By Proposition 2.5, $P \triangleleft G$. If p does not divide the order of G/P , then for any maximal subgroup $X/P \in \Lambda(G/P)$ it follows that $X \in \Xi_p(G)$. Consequently $L(G/P) = S_p(G/P)$. Since for any group X , the subgroup $L(X)$ is supersolvable ([3], a published proof appears in [5, Theorem 3]), the result now follows.

Case 2. p does not divide the order of $S_p(G)$. If q is the largest prime divisor of the order of $S_p(G)$, then for any Sylow q -subgroup Q of $S_p(G)$ one has by Proposition 2.5 that $Q \triangleleft G$. The result now follows by using inductive arguments and Lemma 2.3.

Theorem 2.7 [9, Theorem 11]. *Let G be a group and p, q be two distinct primes dividing the order of G , one of them being the largest prime divisor of the order of G . Then $S_p(G) \cap S_q(G)$ is supersolvable.*

Lemma 2.8 [1, Lemma 3]. *If G is a group with a maximal core-free subgroup, then the following are equivalent:*

- (i) *There exists a unique minimal normal subgroup of G and there exists a common prime-divisor of the indices of all maximal core-free subgroups of G .*
- (ii) *There exists a nontrivial solvable normal subgroup of G .*
- (iii) *The indices of all maximal core-free subgroup of G are powers of a unique prime.*

3. SOME SOLVABILITY CONDITIONS

It was stated by Deskins [6, 2.5] that a group G is solvable if and only if $\eta(G : M) = [G : M]$ for every maximal subgroup M of G (a detailed proof appears in [2]). In [12, Theorem 2.3] we extended this by proving that a group is solvable if and only if the hypothesis is satisfied by only the maximal subgroups of composite indices. We now extend this result further by considering an even smaller class of maximal subgroups.

Theorem 3.1. *For a group G consider the family $\Xi_p(G)$ where p is the largest prime dividing the order of G . Then G is solvable if and only if $\eta(G : M) = [G : M]$ for each M in $\Xi_p(G)$.*

Proof. \Leftarrow : If $\Xi_p(G)$ is empty, then by definition $S_p(G) = G$ and so by Theorem 2.4 (i) it follows that G is solvable. Now, assume that $\Xi_p(G)$ nonempty. Let H be a minimal counterexample to the assertion. Obviously, H is not solvable. Let q be the largest prime dividing the order of H . If $\Xi_q(H)$ is empty, then as above H is solvable, a contradiction. If H is simple, then by the hypothesis $\eta(H : M) = o(H) = [H : M]$ for any $M \in \Xi_p(H)$, a contradiction. Let N be a minimal normal subgroup of H . If q divides the order of H/N , then for a maximal subgroup L/N in H/N of composite index it follows that L is of composite index in H and further q does not divide $[H : L]$. By hypothesis $\eta(H : L) = [H : L]$ and so using Lemma 2.1 we have that

$$\eta(H/N : L/N) = [H/N : L/N]$$

and since this equality holds for arbitrary maximal subgroups of H/N of composite index, by [12, Theorem 2.3] mentioned above it follows that H/N is solvable. On the other hand if q divides the order of H/N then for any maximal subgroup X/N of H/N which is of composite index in H/N and such that $[H/N : X/N]_q = 1$, one has that X is a maximal subgroup of H of composite index and $[H : X]_q = 1$. Now, by using the hypothesis and Lemma 2.1 as before, one obtains that H/N is solvable.

If N_1 and N_2 are two distinct, minimal normal subgroups of H then it follows from above that H/N_1 and H/N_2 are both solvable and consequently $H/N_1 \cap N_2 \cong H$ is solvable, a contradiction. We may therefore assume that N is the unique minimal normal subgroup of H . If $N \subseteq \Phi_q(H)$, then N is solvable using the fact that $\Phi_q(H)$ is solvable [9, Theorem 7(i)]. It then follows that H is solvable, a contradiction. If $N \not\subseteq \Phi_q(H)$ then $H = YN$ for some $Y \in \Upsilon_q(H)$. If $[H : Y] = r$, a prime, then by considering the permutation representation of H on the right cosets of Y and using the fact that Y is core-free, one gets that $o(H)$ divides $r!$, a contradiction to the fact that q is the largest prime dividing the order of H . Thus $[H : Y]$ is composite and so $Y \in \Lambda(H)$. By the hypothesis,

$$\eta(H : Y) = o(N) = [H : Y].$$

Hence q does not divide the order of N . Let M be any core-free maximal subgroup. Then $H = MN$ and $[H : M]_q = 1$. If $[H : M] = t$, a prime, then once again by taking the permutation representation on cosets it follows that $o(H)$ divides $t!$, a contradiction. Consequently $[H : M]$ is a composite number. Thus it follows that $M \in \Lambda(H)$ and by hypothesis

$$\eta(H : M) = o(N) = [H : M].$$

Thus H has a unique minimal normal subgroup and there is a common divisor of the indices of all the core-free maximal subgroups of H . Therefore by Lemma 2.8, N is solvable. This however now implies that H is solvable, a contradiction. Hence the assertion now follows.

The converse holds trivially.

Remark. It might be tempting to conjecture that for a group G and a given prime q , if for each $M \in \Xi_q(G)$ one has that $\eta(G : M)_q = [G : M]_q$ then G is q -solvable. However, this is not true as can be seen by taking G to be PSL(2, 7) and choosing $q = 2$.

Proposition 3.2. *Let p be the largest prime dividing the order of a group G . Then G is solvable if $\eta[G : M]$ is square-free for each M in $\Xi_p(G)$.*

Proof. By using Lemma 2.2, the hypothesis implies that for each M in $\Xi_p(G)$, $\eta(G : M) = [G : M]$. The result now follows using Theorem 3.1.

Corollary 3.3. *Let p be the largest prime dividing the order of a group G . If $\eta[G : M]$ is square-free for each M in $\Upsilon_p(G)$ then G is solvable.*

The following is an analog of Theorem 3.1 and we omit the proof which is similar to the proof of Theorem 3.1.

Theorem 3.4. *Let G be a group and p any prime. Then G is solvable if and only if $\eta(G : M) = [G : M]$ for each M in $\Upsilon_p(G)$.*

We now consider a variation of Theorem 3.1.

Theorem 3.6. *Let p be the largest prime dividing the order of a group G and assume that G is p -solvable. Then G is solvable if and only if the following condition*

$$(*) \quad \eta(G : M)_{p'} = [G : M]_{p'}$$

holds for each M in $\Xi_p(G)$.

Proof. If G is solvable then by [12, Theorem 2.3], $\eta(G : X) = [G : X]$ for each maximal subgroup M of G and so the condition (*) follows trivially. To prove the converse, let G be a minimal counterexample to the assertion. If q is the largest prime dividing $o(G)$, then G is q -solvable but not solvable. If $\Xi_q(G)$ is empty, then $S_q(G) = G$ and so by Theorem 2.4(i) G is solvable, a contradiction. So we may assume that $\Xi_q(G)$ is nonempty. Now, G cannot be simple. For, otherwise for any X in $\Xi_p(G)$ one gets by the hypothesis (*) that

$$\eta(G : X)_{q'} = o(G)_{q'} = [G : X]_{q'}$$

and so X is a Sylow q -subgroup of G . Thus X is a maximal nilpotent subgroup of odd order (we may assume that q is odd for if $q = 2$ then G is a 2-group and so is solvable, a contradiction). Now, by using a result of Thompson ([14]; also see [7, Satz IV, p. 445]), one obtains that G is solvable, a contradiction.

Let N be a minimal normal subgroup of G , Then N is either, a q' -group or, an elementary Abelian q -group. We now prove that G/N is solvable. We distinguish two cases.

Case 1. q does not divide $o(G/N)$. Let X/N be any maximal subgroup of G/N of composite index. It is then easy to see that X belongs to $\Xi_p(G)$ and

so by the hypothesis

$$\eta(G : X)_{q'} = [G : X]_{q'}$$

so that $\eta(G/N : X/N) = [G/N : X/N]$ for each maximal subgroup X/N of G/N of composite index. By [12, Theorem 2.3] (quoted at the beginning of this section), it follows that G/N is solvable.

Case 2. $q|o(G/N)$. If Y/N belongs to $\Xi_q(G/N)$, then clearly Y belongs to $\Xi_q(G)$ and so by applying the hypothesis it follows that

$$\eta(G/N : Y/N)_{q'} = [G/N : Y/N]_{q'}$$

and so as in Case 1 it follows that G/N is solvable.

If N_1 and N_2 are two distinct minimal normal subgroups of G/N , then it follows by the above discussion that both G/N_1 and G/N_2 are solvable and consequently G is solvable, a contradiction. Therefore, we now assume that N is the unique minimal normal subgroup of G . If N is a q -group then N is solvable, so it follows that G is solvable, a contradiction. Consider now the case when N is a q' -group. If $N \subseteq \Phi_q(G)$ then $\Phi_q(G)$ being solvable [9, Theorem 7(i)], N is solvable and so G is solvable, a contradiction. Now, suppose that $N \not\subseteq \Phi_q(G)$. Then for some L in $\Upsilon_q(G)$, $G = LN$. As in the proof of Theorem 3.1 one can show that $[G : L]$ is composite. Thus $L \in \Xi_p(G)$. Since N is a q' -group it now follows by applying the hypothesis that

$$\eta(G : L) = o(N) = [G : L].$$

If M is any other core-free maximal subgroup of G , then $G = MN$ and since N is a q' -group, $[G : M]_q = 1$. Applying Lemma 2.8 it now follows that N is solvable and therefore G is solvable, a contradiction. Hence the result follows.

In [8, Theorem 4] it was proved that a group G is solvable if and only if for any two maximal subgroups M_1 and M_2 of G , one has that $\eta(G : M_1) = \eta(G : M_2)$ implies that $[G : M_1] = [G : M_2]$. We now give an extension of this result.

Theorem 3.7. *Let G be a p -solvable group where p is an arbitrary prime. Then G is solvable if and only if for any M_1 and M_2 in $\Upsilon_p(G)$, one has that $\eta(G : M_1) = \eta(G : M_2)$ implies that $[G : M_1] = [G : M_2]$.*

Proof. Assume the given hypothesis, and let N be a minimal normal subgroup of G . We use induction on the order of G . Using Lemma 2.1 it is easy to see that the hypothesis holds for G/N , so by applying induction it follows that G/N is solvable. Now N is either a p -group or a p' -group. If N is a p -group, then it now follows that G is solvable. Now suppose that N is a p' -group. As in the proof of Theorem 3.1, one may now suppose that N is the unique minimal normal subgroup of G . If $N \subseteq \Phi_p(G)$, then N is solvable and so the result follows. If $N \not\subseteq \Phi_p(G)$, then $G = XN$ for some X in $\Upsilon_p(G)$. It is now easy to see that if L is a core-free maximal subgroup of G , then $[G : L] = [G : X]$. Thus there is a common prime divisor of the indices of all

the core-free maximal subgroups of G . Hence by Lemma 2.8, N is solvable and therefore G is solvable. The converse holds trivially.

4. SOME SUPERSOLVABILITY CONDITIONS

Theorem 4.1. *Let p be the largest prime dividing the order of a group G and suppose that for each maximal subgroup of G , $[G : M]_p = 1$ or p . If for every M in $\Xi_p(G)$ one has that $\eta(G : M)$ is square-free, then G is supersolvable.*

Remark. The condition $[G : M]_p = 1$ or p , is satisfied by $\text{PSL}(2, 7)$ for $p = 7$. However, it is easy to see in this case that for each maximal subgroup M one has that $\eta(G : M) = 168$ which is not square free.

Proof. As in the proof of Theorem 3.1, one may assume that $\Xi_p(G)$ is nonempty and further that G is not simple. Let G be a minimal counterexample. Then G is not supersolvable and G is not simple. Arguing as in the proof of Theorem 3.1, one obtains that G has a unique minimal normal subgroup N and further that G/N is supersolvable.

Let q be the largest prime dividing the order of G . If $N \notin \Upsilon_q(G)$ then $G = YN$ for some $Y \in \Upsilon_q(G)$. Now if $[G : Y]$ is a prime r , say, then by considering the permutation representation of G on the cosets of Y and using the fact that Y is core-free, one obtains that $o(G)$ divides $r!$, a contradiction to the fact that q is the largest prime dividing $o(G)$. So, $N \in \Upsilon_q(G)$. Consequently, N is an elementary Abelian t -group for some prime t . If $N \not\subseteq \Phi(G)$, then for some maximal subgroup M of G one has that $G = MN$ and $M \cap N = \langle 1 \rangle$. We now claim that N is cyclic. For, if $[G : M]$ is not composite then $[G : M] = o(N)$ is a prime, and so N is cyclic. Suppose now that $[G : M]$ is composite. If $t \neq q$ then $[G : M]_q = o(N)_q = 1$ and so $M \in \Xi_q(G)$. Consequently, by using the hypothesis and also Lemma 2.2, it follows that $\eta(G : M) = [G : M]$ is square-free and so N is cyclic. On the other hand, if $t = q$ then $[G : M]_q = q$, and so N is cyclic. Hence in all cases it follows that N is cyclic and so now G/N is supersolvable implies that G is supersolvable, a contradiction. Therefore, we now assume that $N \subseteq \Phi(G)$ and it follows that $G/\Phi(G)$ is supersolvable and consequently G is supersolvable, a contradiction. Hence the theorem now follows.

In [8, Corollary 2] it was proved that a group G is supersolvable if and only if $\eta(G : M)$ is square-free for each maximal subgroup of G . This was extended in [12, Theorem 3.2] to the case when the hypothesis is only satisfied by maximal subgroups of composite index. The next two result are further extensions of this theorem.

Proposition 4.2. *If $\eta[G : M]$ is square-free for each M in $\Xi_p(G)$ where G is a group with p the largest prime dividing $o(G)$, then G is Sylow tower group of supersolvable type.*

Proof. Using Lemma 2.2 we get that for each M satisfying the hypothesis, the order of the chief factor H/K by a minimal supplement H of M is

square-free. It is well known that a group of square-free order is supersolvable. Thus H/K is supersolvable and moreover, is clearly elementary Abelian. Consequently, $o(H/K) = \eta(G : M)$ which is a prime. Thus $[G : M]$ is a prime, a contradiction. It follows therefore that $\Xi_p(G)$ is empty. Hence $G = S_p(G)$ and the result follows using Proposition 2.6.

Proposition 4.3. *A group G is supersolvable if and only if $\eta(G : M)$ is square-free for every maximal subgroup M of G which belongs to $\Xi_p(G)$ or $\Xi_q(G)$ where p and q are two distinct prime divisors of $o(G)$, p being the largest prime divisor of $o(G)$.*

Proof. If G is supersolvable, then the assertion holds since by a well-known result of Huppert [7, 9.5] every maximal subgroup of G is of prime index. To prove the converse, we first obtain by arguing as in the proof of Proposition 4.2 that $G = S_p(G)$. Further, for any M belonging to $\Xi_q(G)$, it follows by arguing as in the proof of Proposition 4.2 that $[G : M] = \eta(G : M)$ is a prime, a contradiction. Therefore, $\Xi_q(G)$ must be empty and consequently $S_q(G) = G$. Thus we now have that $G = S_p(G) = S_q(G)$. Hence by Theorem 2.7 it follows that G is supersolvable.

REFERENCES

1. R. Baer, *Classes of finite groups and their properties*, Illinois J. Math. **1** (1957), 115–187.
2. J. C. Beidleman and A. E. Spencer, *The normal index of maximal subgroups in finite groups*, Illinois J. Math. **16** (1972), 95–101.
3. H. C. Bhatia, *A generalized Frattini subgroup of a finite group*. Ph.D. thesis, Michigan State Univ., East Lansing, 1972.
4. P. Bhattacharya and N. P. Mukherjee, *A family of maximal subgroups containing the Sylow subgroups and some solvability conditions*, Arch. Math. **45** (1985), 390–397.
5. —, *On the intersection of a class of maximal subgroups of a finite group*, II, J. Pure Appl. Algebra **42** (1986), 117–124.
6. W. E. Deskins, *On maximal subgroups*, Proc. Sympos. Pure Math., vol. 1, Amer. Math. Soc., Providence, R.I., 1959, pp. 100–104.
7. B. Huppert, *Endliche Gruppen*, I. Berlin 1967.
8. N. P. Mukherjee, *A note on normal index and maximal subgroups in finite groups*, Illinois J. Math. **75** (1975), 173–178.
9. N. P. Mukherjee and P. Bhattacharya, *On the intersection of a class of maximal subgroups of a finite group*, Canad. J. Math. **39** (1987), 603–611.
10. —, *A generalized Frattini subgroup of a finite group*, Internat. J. Math. Math. Sci. (to appear.)
11. —, *On the intersection of a family of maximal subgroups containing the Sylow subgroups of a finite group*, Canad. J. Math. **40** (1988), 352–359.
12. —, *On the normal index of a finite group*, Pacific J. Math. **132** (1988), 143–149.
13. L. R. Nyhoff, *The influence on a group of the cofactors and subcofactors of its subgroups*, Trans. Amer. Math. Soc. **154** (1971), 459–491.
14. J. G. Thompson, *Finite groups with fixed-point free automorphisms of prime order*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 578–581.

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