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SIEVED ORTHOGONAL POLYNOMIALS AND DISCRETE MEASURES WITH JUMPS DENSE IN AN INTERVAL

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ABSTRACT. We investigate particular classes of sieved Jacobi polynomials for which the weight function vanishes at the zeros of a Chebyshev polynomial of the first kind. These polynomials are then used to give a proof, using only orthogonal polynomials on $[-1,1]$, that the discrete orthogonal polynomials introduced by Lubinsky have converging recurrence coefficients. We construct similar discrete measures with jumps dense in $[-1,1]$ and use sieved ultraspherical polynomials to show that their recurrence coefficients converge.

1. INTRODUCTION

Recently Lubinsky [L] gave a class of discrete measures on $[-1,1]$ for which the orthogonal polynomials have converging recurrence coefficients. Rakhmanov [R1, R2] (see also [MNT]) has shown that the recurrence coefficients for orthogonal polynomials on $[-1,1]$ converge whenever the orthogonality measure has an absolutely continuous part for which the Radon-Nikodym derivative is positive almost everywhere in $[-1,1]$. Lubinsky's example therefore shows that Rakhmanov's condition is not necessary. Random examples of this were known before [DSS] but are not very transparent. This led to the problem of finding nonrandom examples of such discrete measures. The class of orthogonal polynomials with converging recurrence coefficients is denoted by $M(a,b)$ and a detailed investigation of it is given by Nevai [N1]. Lubinsky used orthogonal polynomials on the unit circle to show that the recurrence coefficients in his discrete example converge. We will show that it is possible to prove this without relying on orthogonal polynomials on the unit circle. We will make essential use of sieved orthogonal polynomials. Such orthogonal polynomials have become a major topic in the theory of orthogonal polynomials (Al-Salam, Allaway, and Askey [AAA], Ismall [I1, I3], Charris and Ismail [CI], Geronimo and Van Assche [GVA]).

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In §2 we will investigate sieved orthogonal polynomials with a weight function that vanishes at the zeros of a Chebyshev polynomial of the first kind. We also list some properties of special sieved ultraspherical polynomials. In §3 we show how these sieved orthogonal polynomials can be used in proving that Lubinsky's polynomials have converging recurrence coefficients. We also give two more classes of discrete measures on $[-1,1]$ for which the orthogonal polynomials have converging recurrence coefficients. These new measures are similar to those given by Lubinsky but are in fact easier to handle.

2. Some sieved orthogonal polynomials

In this section we investigate the orthonormal polynomials $\{p_n(x;\sigma_m): n = 0,1,\dots\}$ with weight function

$$\sigma_m(x) = \frac{2}{\pi}\frac{T_m^2(x)}{\sqrt{1-x^2}}, \qquad -1 < x < 1, \tag{2.1}$$

where T_m is the mth degree Chebyshev polynomial of the first kind:

$$T_m(\cos\theta) = \cos m\theta .$$

Like all orthogonal polynomials on the real line with respect to an even weight function, these orthogonal polynomials satisfy a three term recurrence relation:

(2.2)

$$xp_n(x;\sigma_m) = a_{n+1}(\sigma_m)p_{n+1}(x;\sigma_m) + a_n(\sigma_m)p_{n-1}(x;\sigma_m), \qquad n = 0,1,2,\dots$$

with $a_n(\sigma_m) > 0$ $(n = 1,2,\dots)$, $p_0(x;\sigma_m) = 1$ and $p_{-1}(x;\sigma_m) = 0$. Our aim is to find explicit expressions for $p_n(x;\sigma_m)$ and the recurrence coefficients $a_n(\sigma_m)$.

We will begin by investigating the easiest case with $m = 1$ and then handle the general case.

Lemma 1. *The orthogonal polynomials $\{p_n(x;\sigma_1): n = 0,1,\dots\}$ are explicitly given by*

$$\begin{aligned} xp_{2n}(x;\sigma_1) &= T_{2n+1}(x), \\ 2x^2 p_{2n+1}(x;\sigma_1) &= \left\{\frac{2n+1}{2n+3}\right\}^{1/2} T_{2n+3}(x) + \left\{\frac{2n+3}{2n+1}\right\}^{1/2} T_{2n+1}(x), \end{aligned} \tag{2.3}$$

where $\{T_n(x): n = 0,1,\dots\}$ are the Chebyshev polynomials of the first kind. The recurrence coefficients are

$$\begin{aligned} &a_1(\sigma_1) = \{3/4\}^{1/2}, \\ &a_{2n}(\sigma_1) = \frac{1}{2}\left\{\frac{2n-1}{2n+1}\right\}^{1/2}; \qquad a_{2n+1}(\sigma_1) = \frac{1}{2}\left\{\frac{2n+3}{2n+1}\right\}^{1/2}, n > 0. \end{aligned} \tag{2.4}$$

Proof. An easy exercise on symmetric moment functionals ([C], Theorem 8.1 on p. 41 and Ex. 8.6 on p. 44) gives

$$p_{2n}(x;\sigma_1) = 2^{2n-1} P_n^{(-1/2,1/2)}(2x^2-1)\Big/\binom{2n-1}{n},$$

$$p_{2n+1}(x;\sigma_1) = 2^{2n+1}\left\{\frac{2n+1}{2n+3}\right\}^{1/2} x P_n^{(-1/2,3/2)}(2x^2-1)\Big/\binom{2n+1}{n}.$$

The normalizing constants here are different than those in Chihara because we consider orthonormal polynomials whereas Chihara considers monic polynomials. One can rewrite these Jacobi polynomials in terms of Chebyshev polynomials by using standard quadratic transformation formulas ([S], Theorem 4.1 on p. 59 and Equation (4.5.4) on p. 71), which lead to (2.3). The recurrence coefficients $\{a_n(\sigma_1)\colon n=1,2,\dots\}$ satisfy ([C] Theorem 9.1 on p. 46)

$$a_1^2(\sigma_1) = \tfrac{3}{4},$$

$$\begin{cases} a_{2n-1}(\sigma_1)a_{2n}(\sigma_1) = \tfrac{1}{4} \\ a_{2n}^2(\sigma_1) + a_{2n+1}^2(\sigma_1) = \tfrac{1}{2} \end{cases} \qquad n=1,2,\dots$$

and one easily checks that (2.4) is the only solution of these nonlinear recurrence formulas. □

Notice that

$$2x^2 p_{2n}(x;\sigma_1) = 2xT_{2n+1}(x) = T_{2n}(x) + T_{2n+2}(x),$$

so that

$$(2.5)\quad 2x^2 p_n(x;\sigma_1) = \left\{\frac{n+1-(-1)^n}{n}\right\}^{1/2} T_n(x) + \left\{\frac{n}{n+1-(-1)^n}\right\}^{1/2} T_{n+2}(x)$$

holds for every $n>0$.

We can now prove the results for general m:

Theorem 1. *Let m be a positive integer. Then the orthogonal polynomials $\{p_n(x;\sigma_m) = \gamma_n(\sigma_m)x^n + \cdots : n=0,1,\dots\}$ are given by*

(2.6)

$$p_{nm}(x;\sigma_m) = p_n(T_m(x);\sigma_1),$$

$$2T_m^2(x)p_{nm+j}(x;\sigma_m) = \left\{\frac{n+1}{n+2}\right\}^{1/2}\left\{\frac{n+2}{n+1}T_{nm+j}(x) + \frac{(-1)^n}{n+1}T_{(n+2)m-j}(x) + T_{(n+2)m+j}(x)\right\} \qquad j=1,\dots,m-1.$$

The leading coefficients $\{\gamma_n(\sigma_m)\colon n=1,2,\dots\}$ are given by

$$\gamma_0(\sigma_m) = 1,$$

$$(2.7)\quad \gamma_{mn}(\sigma_m) = 2^{mn}\left\{\frac{n}{n+1-(-1)^n}\right\}^{1/2}, \qquad n=1,2,\dots$$

$$\gamma_{mn+j}(\sigma_m) = 2^{mn+j}\left\{\frac{n+1}{n+2}\right\}^{1/2}, \qquad n=0,1,\dots; \qquad 1\le j\le m-1.$$

If $x \in [-1, 1]$ *then*

$$|T_m(x)p_n(x;\sigma_m)| \le 2, \qquad n = 0, 1, \dots. \tag{2.8}$$

Proof. We will use the techniques in [GVA] (especially §VI) to map the weight $\sigma_1(x) = (2/\pi)x^2/\sqrt{1-x^2}$ on $[-1, 1]$ to the weight $\sigma(x) = |U_{m-1}|\sigma_1(T_m(x))$ by means of the polynomial transformation using the mapping $T(x) = T_m(x)$. One easily verifies that $\sigma(x) = \sigma_m(x)$, which is indeed the weight we want to consider. According to the results in [GVA] one has

$$p_{nm}(x;\sigma_m) = p_n(T_m(x);\sigma_1),$$

$$U_{m-1}(x)p_{nm+j}(x;\sigma_m) = a_{n+1}(\sigma_1)\left\{\frac{1}{a_{nm+m}(\sigma_m)}U_{m-j-1}(x)p_n(T_m(x);\sigma_1)\right.$$
$$\left.+\frac{1}{a_{nm+1}(\sigma_m)}U_{j-1}(x)p_{n+1}(T_m(x);\sigma_1)\right\}, \qquad j = 1, \dots, m-1,$$

where $\{U_n(x): n = 0, 1, \dots\}$ are the Chebyshev polynomials of the second kind:

$$U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}.$$

The recurrence coefficients $\{a_n(\sigma_m): n = 1, 2, \dots\}$ are given by ([GVA] Theorem 12)

$$a_{mn+j}(\sigma_m) = \tfrac{1}{2}, \qquad j = 2, \dots, m-1,$$
$$a^2_{nm}(\sigma_m) = \frac{a_n(\sigma_1)}{2}\frac{p_{n-1}(1;\sigma_1)}{p_n(1;\sigma_1)}, \qquad a^2_{nm+1}(\sigma_m) = \tfrac{1}{2} - a^2_{nm}(\sigma_m).$$

From Lemma 1 one easily finds

$$p_{2n}(1;\sigma_1) = 1; \qquad p_{2n+1}(1;\sigma_1) = \left(\left\{\frac{2n+3}{2n+1}\right\}^{1/2} + \left\{\frac{2n+1}{2n+3}\right\}^{1/2}\right)/2$$

and together with (2.4) and $\gamma_n(\sigma_m) = \{a_1(\sigma_m)a_2(\sigma_m)\cdots a_n(\sigma_m)\}^{-1}$ this gives (2.7) when $m > 1$ (for $m = 1$ use Lemma 1). Using (2.5) and the identites

$$T_n(T_m(x)) = T_{nm}(x); \qquad 2U_{m-1}(x)T_n(x) = U_{m+n-1}(x) - U_{n-m-1}(x) \tag{2.9}$$

where $n > m$, one finds (after some lengthy but straightforward algebra)

$$2T^2_m(x)U_{m-1}(x)p_{nm+j}(x;\sigma_m)$$
$$= a_{mn+m}(\sigma_m)\left\{\frac{n+2-(-1)^n}{n+3}\right\}^{1/2}\left\{\frac{n+2}{n+1}\{U_{(n+1)m+j-1}(x) - U_{(n-1)m+j-1}(x)\}\right.$$
$$+\frac{(-1)^n}{n+1}\{U_{(n+3)m-j-1}(x) - U_{(n+1)m-j-1}(x)\}$$
$$\left.+\{U_{(n+3)m+j-1}(x) - U_{(n+1)m+j-1}(x)\}\right\},$$

from which one finds (2.6) by means of (2.9). From (2.6) one also finds

$$2T_m(x)p_{nm+j}(x;\sigma_m) = \left\{\frac{n+1}{n+2}\right\}^{1/2} \left\{ \frac{T_{nm+j}(x) + T_{(n+2)m+j}(x)}{T_m(x)} + \frac{1}{n+1}\frac{T_{nm+j}(x) + (-1)^n T_{(n+2)m-j}(x)}{T_m(x)} \right\}.$$

Now

$$\frac{T_{nm+j}(x) + T_{(n+2)m+j}(x)}{T_m(x)} = 2T_{(n+1)m+j}(x),$$

$$\frac{T_{nm+j}(x) + (-1)^n T_{(n+2)m-j}(x)}{T_m(x)} = \begin{cases} 2T_{m-j}(x)T_{n+1}(T_m x))/T_m(x) & (n \text{ even}) \\ -2U_{m-1}(x)U_{m-j-1}(x)U_n(T_m(x))/T_m(x) & (n \text{ odd}) \end{cases}$$

and by means of the inequalities

$$\begin{aligned} &|T_{n+1}(x)/x| \le n+1 \quad (n \text{ even}) \\ &|U_n(x)/x| \le n+1 \quad (n \text{ odd}) \end{aligned} \qquad -1 \le x \le 1$$

one finds

$$|T_m(x)p_{nm+j}(x)| \le 2\left\{\frac{n+1}{n+2}\right\}^{1/2} < 2$$

which holds for $-1 \le x \le 1$, $j = 0, 1, \cdots, m-1$ and $n \ge 0$ (one easily verifies that the inequality holds also for $j = 0$). □

Let w_m be the weight on $[-1, 1]$ given by

$$w_m(x) = \frac{2}{\pi}U_{m-1}^2(x)\sqrt{1-x^2}$$

where U_n are the Chebyshev polynomials of the second kind. The orthonormal polynomials $\{p_n(x;w_m): n = 0, 1, 2, \dots\}$ for this weight are a particular case of sieved ultraspherical polynomials of the first kind [AAA]:

$$p_n(x;w_m) = c_n(x;m)/H_n$$

where $\{c_n(x;m)\}$ are the polynomials studied in [AAA] (with $\lambda = 1$) and

$$H_{nm} = \frac{2}{n+1}, \qquad n = 1, 2, \dots$$

$$H_{nm+j} = \frac{2}{\sqrt{(n+1)(n+2)}}, \qquad n = 0, 1, 2, \dots; \quad j = 1, \dots, m-1.$$

The weight w_m can be obtained from the weight w_1 (which is the weight function for Chebyshev polynomials of the second kind) by the polynomial mapping $T(x) = T_m(x)$. By Theorem 12 in [GVA] one therefore finds

$$a_{nm+j}(w_m) = \tfrac{1}{2}, \qquad j = 2, \dots, m-1; \qquad n = 0, 1, \dots$$

$$a_{nm}(w_m) = \left\{\frac{n}{n+1}\right\}^{1/2}/2 \quad (n \ge 1), \; a_{nm+1}(w_m) = \left\{\frac{n+2}{n+1}\right\}^{1/2}/2 \quad (n \ge 0)$$

so that for every $n \geq 0$

$$\gamma_{nm}(w_m) = 2^{nm}, \qquad \gamma_{nm+j}(w_m) = 2^{mn+j}\left\{\frac{n+1}{n+2}\right\}^{1/2}, \qquad (1 \leq j \leq m-1),$$

and the orthogonal polynomials are given by

$$\begin{aligned} p_{nm}(x;w_m) &= U_n(T_m(x)), \qquad n = 0, 1, \dots \\ U_{m-1}(x)p_{nm+j}(x;w_m) &= \left\{\frac{n+2}{n+1}\right\}^{1/2} U_{m-j-1}(x)U_n(T_m(x)) \\ &\quad + \left\{\frac{n+1}{n+2}\right\}^{1/2} U_{j-1}(x)U_{n+1}(T_m(x)). \end{aligned}$$

If one uses the identities

$$(2.10) \qquad \begin{aligned} &U_{m-1}(x)U_{n-1}(T_m(x)) = U_{nm-1}(x), \\ &-2(1-x^2)U_{n-1}(x)U_{m-1}(x) = T_{n+m}(x) - T_{n-m}(x), \qquad n \geq m \end{aligned}$$

then one finds

$$\begin{aligned} 2(1-x^2)U_{m-1}^2(x)p_{nm+j}(x;w_m) &= \left\{\frac{n+1}{n+2}\right\}^{1/2} \left\{\frac{n+2}{n+1}T_{nm+j}(x)\right. \\ &\quad \left. -\frac{1}{n+1}T_{(n+2)m-j}(x) - T_{(n+2)m+j}(x)\right\} \\ &= \left\{\frac{n+1}{n+2}\right\}^{1/2} \left\{\{T_{nm+j}(x) - T_{(n+2)m+j}(x)\}\right. \\ &\quad \left. + \frac{1}{n+1}\{T_{nm+j}(x) - T_{(n+2)m-j}(x)\}\right\}. \end{aligned}$$

Now use (2.10) again and the inequalities $(-1 \leq x \leq 1)$

$$|U_n(x)| \leq n+1, \qquad |T_n(x)| \leq 1, \qquad |\sqrt{1-x^2}U_n(x)| \leq 1.$$

Then one finds

$$(2.11) \qquad |\sqrt{1-x^2}U_{m-1}(x)p_{nm+j}(x;w_m)| \leq 2\left\{\frac{n+1}{n+2}\right\}^{1/2} < 2$$

which holds for $-1 \leq x \leq 1$, $j = 0, 1, \dots, m-1$ and every n (for $j = 0$ the upper bound can be replaced by 1). Notice that inequalities (2.8) and (2.11) are special cases of more general inequalities for orthogonal polynomials ([N2], Equation (16)).

3. Discrete measures with jumps dense in $[-1, 1]$

The discrete measures introduced by Lubinsky are given by

$$\mu = \sum_{k=1}^{\infty} \eta_k \mu_{3^k},$$

where μ_n are discrete measures with jumps at the zeros of $T_n(x)$:

$$\int f(x)\,d\mu_n(x) = \frac{1}{n}\sum_{j=1}^{n} f\left(\cos\frac{2j-1}{2n}\pi\right)$$

for every continuous function f on $[-1,1]$, $\{\eta_k : k = 1, 2, \ldots\}$ is a sequence of positive real numbers such that

$$\sum_{k=1}^{\infty} \eta_k = 1.$$

By the Gauss-Jacobi quadrature formula for Chebyshev polynomials of the first kind ([S], pp. 47–49) one has

$$\int p(x)\,d\mu_n(x) = \frac{1}{\pi}\int_{-1}^{1} p(x)\frac{dx}{\sqrt{1-x^2}}$$

for every polynomial p of degree at most $2n-1$. This shows that μ is a positive measure on $[-1,1]$ with total mass one. Moreover the measure μ is discrete and the jumps are dense in $[-1,1]$.

Theorem 2. *Let $\{p_n(x;\mu): n = 0, 1, 2, \ldots\}$ be the orthogonal polynomials with orthogonality measure μ, and denote the leading coefficient of $p_n(x;\mu)$ by $\gamma_n(\mu)$, then for $m = 3^{j-1} \le 3^{l-1} \le n < 3^l$*

$$2^{2n-1}\frac{n-m}{n+m}\Big/\left\{8\sum_{k=j}^{l-1}\eta_k + \sum_{k=l}^{\infty}\eta_k\right\} \le \gamma_n^2(\mu) \le 2^{2n-1}\Big/\sum_{k=l}^{\infty}\eta_k. \tag{3.1}$$

If moreover $\eta_l/\sum_{k=l}^{\infty}\eta_k$ tends to zero as $l \to \infty$ then the recurrence coefficients $\{a_n^{(\mu)} = \gamma_{n-1}(\mu)/\gamma_n(\mu): n = 1, 2, \ldots\}$ converge to $\frac{1}{2}$ as $n \to \infty$.

Proof. The upper bound in (3.1) was already given by Lubinsky [L]. For the lower bound we use the well-known extremal property

$$\gamma_n(\mu)^{-2} \le \int q_n^2(x)\,d\mu(x),$$

where q_n is any monic polynomial of degree n with real coefficients and equality holds for $q_n(x) = p_n(x;\mu)/\gamma_n(\mu)$. Let $m = 3^{j-1}$ and choose

$$q_n(x) = 2^{1-m}T_m(x)p_{n-m}(x;\sigma_m)/\gamma_{n-m}(\sigma_m)$$

where $\gamma_n(\sigma_m) = \{a_1(\sigma_m)a_2(\sigma_m)\cdots a_n(\sigma_m)\}^{-1}$ is the leading coefficient of the sieved orthogonal polynomial $p_n(x;\sigma_m)$, then

$$\gamma_n(\mu)^{-2} \le \frac{2^{2-2m}}{\gamma_{n-m}^2(\sigma_m)}\left\{\sum_{k=j}^{l-1}\eta_k\int T_m^2(x)p_{n-m}^2(x;\sigma_m)\,d\mu_{3^k}(x)\right.$$
$$\left. + \sum_{k=l}^{\infty}\eta_k\frac{1}{2}\int_{-1}^{1}p_{n-m}^2(x;\sigma_m)\sigma_m(x)\,dx\right\}.$$

The summation starts from j since T_m vanishes at the zeros of T_{3^k} when $k < j$. In the second sum we have used the Gauss-Jacobi quadrature which holds because $2n \leq 2.3^k - 1$ for every $k \geq l$. The second sum clearly reduces to

$$\frac{1}{2}\sum_{k=l}^{\infty}\eta_k .$$

For the first sum we use (2.8). This gives

$$\gamma_n(\mu)^{-2} \leq 2^{1-2m}\left\{8\sum_{k=j}^{l-1}\eta_k + \sum_{k=l}^{\infty}\eta_k\right\}/\gamma_{n-m}^2(\sigma_m).$$

From (2.7) one finds

$$\gamma_n(\sigma_m)^{-1} \leq 2^{-n}\left\{\frac{n+2m}{n}\right\}^{1/2}, \qquad n = 1, 2, \ldots$$

which gives the lower bound in (3.1).

In order to show that $a_n(\mu)$ converges to $\frac{1}{2}$, we only need to show that

$$\lim_{n\to\infty}\sum_{k=j}^{l-1}\eta_k/\sum_{k=l}^{\infty}\eta_k = 0$$

for an appropriate choice of j. Clearly, by assumption, there exists a positive sequence $\{c_k : k = 1, 2, \ldots\}$ with $c_k \to 0$, such that

$$\eta_n \leq c_n\sum_{k=n}^{\infty}\eta_k .$$

Simple estimation gives

$$\sum_{k=j}^{l-1}\eta_k/\sum_{k=l}^{\infty}\eta_k \leq \sum_{k=j}^{l-1}c_k/\left\{1 - \sum_{k=j}^{l-1}c_k\right\},$$

provided the denominator on the right is positive. Hence if we choose j such that $l - j \to \infty$ but $\sum_{k=j}^{l-1}c_k \to 0$ as $n \to \infty$, then the result follows. □

Next we introduce another class of discrete measures. Let

$$\nu = \sum_{k=1}^{\infty}\eta_k\nu_{2^k},$$

where ν_n is a discrete measure with jumps at the zeros of U_{n-1}:

$$\int f(x)\,d\nu_n(x) = \frac{2}{n}\sum_{j=1}^{n-1} f\left(\cos\frac{j\pi}{n}\right)\sin^2\frac{j\pi}{n}$$

for every continuous function f on $[-1, 1]$, and $\{\eta_k : k = 1, 2, \ldots\}$ is as before. By the Gauss-Jacobi quadrature formula for Chebyshev polynomials of the second kind one now has

$$\int p(x)\,d\nu_n(x) = \frac{2}{\pi}\int_{-1}^{1} p(x)\sqrt{1-x^2}\,dx$$

for every polynomial of degree at most $2n-3$. For these discrete measures we have

Theorem 3. *Let* $\{p_n(x;\nu): n=0,1,2,\dots\}$ *be the orthogonal polynomials for the measure* ν *and denote the leading coefficient of* $p_n(x;\nu)$ *by* $\gamma_n(\nu)$, *then for* $m=2^{j-1}\le 2^{l-1}\le n+1<2^l$

$$2^{2n}\frac{n-m+1}{n+1}\Big/\left\{8\sum_{k=j}^{l-1}\eta_k+\sum_{k=l}^{\infty}\eta_k\right\}\le\gamma_n^2(\nu)\le 2^{2n}\Big/\sum_{k=l}^{\infty}\eta_k. \tag{3.2}$$

If moreover $\eta_l/\sum_{k=l}^{\infty}\eta_k$ *tends to zero as* $l\to\infty$ *then the recurrence coefficients* $a_n(\nu)=\gamma_{n-1}(\nu)/\gamma_n(\nu)$ *for these orthogonal polynomials converge to* $\frac{1}{2}$.

Proof. The upper bound can be proved as in [L]: clearly if $\hat{p}_n(x;\nu)$ is the monic orthogonal polynomial corresponding to ν we have

$$\begin{aligned}\gamma_n(\nu)^{-2}&=\int\hat{p}_n^2(x;\nu)\,d\nu(x)\\&\ge\sum_{k=l}^{\infty}\eta_k\int\hat{p}_n^2(x;\nu)\,d\nu_{2^k}(x)\\&=\frac{2}{\pi}\int_{-1}^{1}\hat{p}_n^2(x;\nu)\sqrt{1-x^2}\,dx\sum_{k=l}^{\infty}\eta_k,\end{aligned}$$

where the last equality follows from the Gauss-Jacobi quadrature which holds because $2n\le 2(2^l-1)-1=2.2^k-3$ for every $k\ge l$. The last integral is minimized when one replaces $\hat{p}_n(x;\nu)$ by $2^{-n}U_n(x)$ and thus we find the upper bound in (3.2). For the lower bound we use

$$\gamma_n(\nu)^{-2}\le\int q_n^2(x)\,d\nu(x)$$

with

$$q_n(x)=2^{1-m}U_{m-1}(x)p_{n-m+1}(x;w_m)/\gamma_{n-m+1}(w_m),$$

where $\gamma_n(w_m)=\{a_1(w_m)a_2(w_m)\cdots a_n(w_m)\}^{-1}$ is the leading coefficient of the sieved ultraspherical polynomial $p_n(x;w_m)$ and $m=2^{j-1}$. This gives

$$\begin{aligned}\gamma_n(\nu)^{-2}\le\frac{2^{2-2m}}{\gamma_{n-m+1}^2(w_m)}&\left\{\sum_{k=j}^{l-1}\eta_k\int U_{m-1}^2(x)p_{n-m+1}^2(x;w_m)\,d\nu_{2^k}(x)\right.\\&\left.+\sum_{k=l}^{\infty}\eta_k\int_{-1}^{1}p_{n-m+1}^2(x;w_m)w_m(x)\,dx\right\}.\end{aligned}$$

The first sum starts at j since U_{m-1} vanishes at the zeros of U_{2^k-1} when $k<j$. In the second sum we have used the Gauss-Jacobi quadrature which holds because $2n\le 2.2^k-3$ when $k\ge l$. The integral in the second sum is

one; for the integral in the first sum we use the inequality (2.11) to find

$$\begin{aligned}\int U_{m-1}^2(x)p_{n-m+1}^2(x;w_m)\,d\nu_{2^k}(x) &= 2^{1-k}\sum_{j=1}^{2^k-1}|U_{m-1}(x)p_{n-m+1}(x)\sqrt{1-x^2}|^2_{x=\cos(j\pi/2^k)} \\ &\le 8.\end{aligned}$$

The lower bound in (3.2) then follows if one takes into account that

$$\gamma_n(w_m)^{-1} \le \left(\frac{n+m}{n}\right)^{1/2} 2^{-n}.$$

The convergence of $a_n(\nu)$ to $\frac{1}{2}$ now follows as in Theorem 2. □

A third class of discrete measures can be given for which exactly the same techniques can be used to prove that the recurrence coefficients converge to $\frac{1}{2}$. These measures are defined by

$$\xi = \sum_{k=1}^{\infty} \eta_k \xi_{2^k},$$

with $\{\eta_k: = 1, 2, \ldots\}$ as before and with ξ_n a discrete measure supported at the extremal points of T_n in $[-1,1]$ (i.e., at the zeros of U_{n-1} and at ± 1):

$$\int f(x)\,d\xi_n(x) = \frac{1}{n}\sum_{j=0}^{n}{}'' f\left(\cos\frac{j\pi}{n}\right),$$

where f is a continuous function and $\sum''$ is a sum in which the first and the last term are divided by two. One easily checks that

$$\int p(x)\,d\xi_n(x) = \frac{1}{\pi}\int_{-1}^{1} p(x)\frac{dx}{\sqrt{1-x^2}}$$

holds for every polynomial of degree at most $2n-1$ (take $p(x) = T_k(x)$ for $k = 0, 1, \ldots, 2n-1$). This is in fact Lobatto's quadrature formula for Chebyshev polynomials of the first kind. We leave the proof that $a_n(\xi)$ converges to $\frac{1}{2}$ to the reader but mention that one can use the sieved ultraspherical polynomials of the second kind (see [AAA]).

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