

## ON TOPOLOGICAL ENTROPY OF GROUP ACTIONS ON $S^1$

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**ABSTRACT.** In this paper, we show that a surface group action on  $S^1$  with a non-zero Euler number has a positive topological entropy. We also show that if a surface group action on  $S^1$  has a Euler number which attains the maximal absolute value in the inequality of Milnor-Wood, then the topological entropy of the action equals the exponential growth rate of the group.

### 1. INTRODUCTION

The definition of the geometric entropy of a foliation was given by Ghys, Langevan and Walczak in [G-L-W] (see also Hurder [Hu]). For the case when the foliation is obtained via the suspension of a group action, it relates to the topological entropy of the group action. In this note, we study the relation between the topological entropy and the homotopical invariant of an action of a surface group.

Let  $G = \text{Homeo}(S^1)$  be the group of all homeomorphisms of  $S^1$  and let  $G^+$  be its subgroup consisting of all orientation preserving ones. Let  $\Sigma_g$  denote a closed oriented surface of genus  $g \geq 2$ . We fix the standard symmetric generating set  $\Gamma_0^g = \{\alpha_i, \beta_i, \alpha_i^{-1}, \beta_i^{-1}; 1 \leq i \leq g\}$  for the fundamental group  $\Gamma^g = \pi_1(\Sigma_g)$  so that  $\Gamma^g$  has the presentation

$$\Gamma^g = \left\langle \alpha_i, \beta_i, 1 \leq i \leq g; \prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \right\rangle.$$

For a given homomorphism  $\phi: \Gamma^g \rightarrow G^+$ ,  $\chi(\phi)$  denotes the Euler number of a circle bundle associated with  $\phi$ . The following inequality is due to Milnor [Mi] and Wood [Wo].

$$|\chi(\phi)| \leq |\chi(\Sigma_g)|,$$

where  $\chi(\Sigma_g)$  is the Euler number of  $\Sigma_g$ .

Let  $\text{gr}(\Gamma^g, \Gamma_0^g)$  denote the exponential growth rate of  $\Gamma^g$  with respect to  $\Gamma_0^g$ . Let  $h(\phi, \Gamma_0^g)$  denote the topological entropy of  $\phi$  with respect to  $\Gamma_0^g$  defined in Section 2.

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Our main results are the following:

**Theorem 1.** *Let  $\phi: \Gamma^g \rightarrow G^+$  be a homomorphism. If  $\chi(\phi) \neq 0$ , then  $h(\phi, \Gamma_0^g) > 0$ .*

**Theorem 2.** *Let  $\phi: \Gamma^g \rightarrow G^+$  be a homomorphism. If  $|\chi(\phi)| = |\chi(\Sigma_g)|$ , then  $h(\phi, \Gamma_0^g) = \text{gr}(\Gamma^g, \Gamma_0^g)$ .*

In Section 2, we give the definition of the topological entropy of a group action and state its fundamental properties. In Section 3 and 4, we will prove Theorem 1 and 2.

## 2. PRELIMINARIES

Let  $\Gamma$  be a finitely generated group with a finite symmetric generating set  $\Gamma_0$  (i.e. if  $\gamma \in \Gamma_0$ , then  $\gamma^{-1} \in \Gamma_0$ ). Let  $(X, d)$  be a compact metric space and  $\phi: \Gamma \rightarrow \text{Homeo}(X)$  a homomorphism. We denote a word norm of  $\Gamma$  with respect to  $\Gamma_0$  by  $\|\cdot\|$ , and set

$$B_n(\Gamma, \Gamma_0) = \{\gamma \in \Gamma; \|\gamma\| \leq n\}.$$

A new metric  $d_n$  for  $X$  is defined by

$$d_n(x, y) = \text{Max}_{\gamma \in B_n} d(\phi(\gamma)x, \phi(\gamma)y), \quad \text{for } x, y \in X.$$

**Definition 1.** Let  $n$  be a natural number and  $\varepsilon$  a positive number.

(1) A subset  $F \subset X$  is called an  $(n, \varepsilon)$ -spanning set if for any  $x \in X$ , there exists a  $y \in F$  such that  $d_n(x, y) \leq \varepsilon$ .

(2) A subset  $E \subset X$  is called an  $(n, \varepsilon)$ -separated set if for two different points  $x, y \in E$ , we have  $d_n(x, y) > \varepsilon$ .

We set

$$s(n, \varepsilon) = \text{Min}\{\#F; F \text{ is an } (n, \varepsilon)\text{-spanning set}\}$$

$$t(n, \varepsilon) = \text{Max}\{\#E; E \text{ is an } (n, \varepsilon)\text{-separated set}\},$$

where  $\#F$  (resp.  $\#E$ ) denotes the cardinality of  $F$  (resp.  $E$ ). Notice that  $s(n, \varepsilon) \leq t(n, \varepsilon) \leq s(n, \varepsilon/2)$ . Then the topological entropy of  $\phi$  with respect to  $\Gamma_0$  is defined by

$$h(\phi, \Gamma_0) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (1/n) \log s(n, \varepsilon)$$

$$= \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (1/n) \log t(n, \varepsilon).$$

The following is easy to prove.

**Proposition 1.** *For given two homomorphisms  $\phi, \psi: \Gamma \rightarrow \text{Homeo}(X)$ , if there exists a continuous surjective map  $f: X \rightarrow X$  such that  $f \circ \phi = \psi \circ f$ , then  $h(\phi) \geq h(\psi)$ .*

In the case  $\phi: Z \rightarrow \text{Homeo}(S^1)$ , the topological entropy of  $\phi$  is always zero (see [Wa, Th. 7.14]). The same argument proves the following:

**Proposition 2.** *Let  $\phi: \Gamma \rightarrow G = \text{Homeo}(S^1)$  be a homomorphism. Then  $h(\phi, \Gamma_0) \leq \text{gr}(\Gamma, \Gamma_0)$ , where  $\text{gr}(\Gamma, \Gamma_0) = \lim_{r \rightarrow \infty} (1/n) \log \#B_n(\Gamma, \Gamma_0)$ .*

Let  $\phi: \Gamma \rightarrow \text{Homeo}(K)$  be a homomorphism, where  $K$  is a one dimensional manifold.

**Definition 2.** An orbit of  $z \in K$  is called a *resilient orbit* if there exist  $\gamma_1, \gamma_2 \in \Gamma$  such that  $\phi(\gamma_1)z \neq z$  and  $\lim_{n \rightarrow \infty} \phi(\gamma_2^n \gamma_1)z = z$ .

**Proposition 3** ([G-L-W]). *If there exists a resilient orbit then the topological entropy of  $\phi$  is positive. If the action is of class  $C^2$ , then the converse is true.*

For the proof, see [G-L-W].

*Remark.* Hector [He] constructed a two generator subgroup of the group of  $C^\infty$ -diffeomorphisms of the interval  $[0, 1]$  which has positive exponential growth. He has shown that it has a subexponential growth orbit which is dense in  $[0, 1]$ . On the other hand, Cantwell and Conlon [C-C] has shown that if the closure of an orbit contains a resilient orbit, then it has positive exponential growth. Hence Hector's group has no resilient orbit. Proposition 3 shows that the topological entropy of the group is zero. This is an example of a group of positive exponential growth with zero topological entropy.

### 3. PROOF OF THEOREM 1

For the proof of Theorem 1, we use a criterion due to Matsumoto [Ma-1] for the vanishing of the pull back of the bounded real Euler class. Let  $\phi: \Gamma \rightarrow G^+$  be a homomorphism and let  $e_{\mathbf{R}}$  be the bounded real Euler class. For more detail of the bounded cohomology and the bounded Euler class, see [Gr], [Gh], [Ma-1], [Ma-2].

**Proposition 4** ([Ma-1]). *The following are equivalent.*

- (1)  $\phi^*(e_{\mathbf{R}}) \neq 0$ .
- (2) *There exist a minimal set  $M$  of  $\phi$  and a  $\gamma \in \Gamma$  such that  $\phi(\gamma)|_M \neq \text{Id}$  and  $\phi(\gamma)$  has a fixed point in  $M$ .*

For the proof, see [Ma-1].

To prove Theorem 1, by Proposition 3, it is sufficient to prove the following:

**Proposition 5.** *Let  $\Gamma$  be a finitely generated group and  $\phi: \Gamma \rightarrow G^+$  a homomorphism. If  $\phi^*(e_{\mathbf{R}}) \neq 0$ , then  $\phi$  has a resilient orbit.*

*Proof.* Suppose that  $\phi^*(e_{\mathbf{R}})$  does not vanish. We set

$$S_\gamma = \{x \in S^1; \phi(\gamma)x \neq x\}.$$

By Proposition 4, there exist a minimal set  $M$  of  $\phi$  and a  $\gamma \in \Gamma$  such that  $S_\gamma \cap M \neq \emptyset$  and  $S_\gamma \neq S^1$ . Let  $x \in S_\gamma \cap M$ . Since  $S_\gamma$  is open, the connected component of  $S_\gamma$  containing  $x$  is an open interval of  $S^1$ . Notice that  $\lim_{n \rightarrow \infty} \phi(\gamma^n)x$  is an endpoint of the interval, which we denote by  $y$ . Since  $M$

is a  $\phi$ -invariant closed set,  $y$  is in  $M$ . Since the  $\phi$ -orbit of  $y$  is dense in  $M$ , the  $\phi$ -orbit of  $y$  intersects the interval. Therefore the orbit of  $y$  is a resilient orbit.

*Remark.* Let  $\Gamma$  be a subgroup of  $G^+$  which is generated by the two homeomorphisms  $z \rightarrow az, z \rightarrow z + b$  ( $0 < |a| < 1, b \neq 0$ ) on  $R \cup \{\infty\} \cong S^1$ . The action of  $\Gamma$  has a positive entropy, since the orbit of 0 is a resilient orbit. On the other hand, the second bounded real cohomology group of  $\Gamma$  vanishes by the amenability of  $\Gamma$  ([Gr]). Therefore, the converse of Proposition 5 is not true.

#### 4. PROOF OF THEOREM 2

If we give  $\Sigma_g$  a Riemann metric of constant negative curvature, then the universal covering space of  $\Sigma_g$  is identified with the Poincare disk  $\Delta$  and  $\Gamma^g = \pi_1(\Sigma_g)$  acts on  $\Delta$  by covering transformations which are isometries. This provides an action of  $\Gamma^g$  on  $S^1 \cong \partial\Delta$ . This action is called a Fuchsian action. In [Go], Goldman has shown that a homomorphism  $\phi: \Gamma^g \rightarrow \text{PSL}(2, \mathbf{R}) \cong \text{Isom}^+(\Delta)$  is a Fuchsian action if and only if  $\chi(\phi) = \chi(\Sigma_g)$ , where  $\text{Isom}^+(\Delta)$  denotes the group of orientation preserving isometries on  $\Delta$ .

To prove Theorem 2, it is sufficient to show it for a Fuchsian action  $\tilde{\phi}$ . For suppose that  $\phi: \Gamma^g \rightarrow G^+$  is a homomorphism such that  $|\chi(\phi)| = |\chi(\Sigma_g)|$ . Since  $|\chi(\tilde{\phi})| = |\chi(\Sigma_g)| = |\chi(\phi)|$ , there exists a semiconjugacy  $f: S^1 \rightarrow S^1$  such that  $f \circ \phi = \tilde{\phi} \circ f$  by Matsumoto [Ma-2] (for the definition of a semiconjugacy, see [Gh], [Ma-1]). Since Fuchsian action is minimal,  $f$  is continuous and surjective ([Gh]), and by Propositions 1 and 2 we have  $h(\tilde{\phi}, \Gamma_0^g) \leq h(\phi, \Gamma_0^g) \leq \text{gr}(\Gamma^g, \Gamma_0^g)$ .

We set

$$M_n = \#(B_n(\Gamma^g, \Gamma_0^g) - B_{n-1}(\Gamma^g, \Gamma_0^g)).$$

Using the fact  $\text{gr}(\Gamma^g, \Gamma_0^g) > 0$ , an easy calculation shows

$$\limsup_{n \rightarrow \infty} (1/n) \log M_n = \text{gr}(\Gamma^g, \Gamma_0^g).$$

We identify  $\Gamma^g$  with a Fuchsian group  $\phi(\Gamma^g)$ . We recall the definition of boundary expansion of a point on  $S^1$  with respect to  $\Gamma^g$  ([B], [S]). For each  $\gamma \in \Gamma_0^g$ , let  $C(\gamma) = \{z \in \Delta; |D\gamma(z)| = 1\}$  be an isometric circle of  $\gamma$ . Then  $\gamma C(\gamma) = C(\gamma^{-1})$  and the  $4g$ -gon  $R$  surrounded by  $C(\gamma)$  ( $\gamma \in \Gamma_0^g$ ) is a fundamental domain of  $\Gamma^g$ . Let the elements of  $\Gamma_0^g$  be labelled by  $\gamma_1, \gamma_2, \dots, \gamma_{4g}$  so that  $C(\gamma_1), \dots, C(\gamma_{4g})$  are in anticlockwise order round  $R$ . Let  $P_i$  and  $Q_i$  be endpoints of  $C(\gamma_i)$ , where  $P_i$  comes before  $Q_i$  in anticlockwise order. Define  $f: \partial\Delta \rightarrow \partial\Delta, f|[P_i, P_{i-1})(x) = \gamma_i x$ . The  $f$ -expansion of  $x \in \partial\Delta$  is the sequence  $x_f = \gamma_{i_0} \gamma_{i_1} \gamma_{i_2} \dots, \gamma_{i_j} \in \Gamma_0^g$ , where  $f^n(x) \in [P_{i_n}, P_{i_{n+1}})$ ,  $n \in \mathbf{N}$ . Let  $\Sigma^+ = \{x_f: x \in \partial\Delta\} \subset \prod_{i=1}^\infty \Gamma_0^g$ . This correspondence  $\partial\Delta \rightarrow \Sigma^+$  is a bijection. Let  $F(\Sigma^+)$  be the finite sequences which occur in  $\Sigma^+$ .

In [S], Series has shown that each element  $\gamma$  in  $\Gamma^g$  has a unique shortest representative  $\omega_\gamma$  in  $F(\Sigma^+)$ . For  $\omega = e_1 e_2 \dots e_k$  in  $F(\Sigma^+)$ , we set  $Z(\omega) = \{x \in \partial\Delta; x_f = e_1 e_2 \dots e_k \dots\}$ . Then  $Z(\omega)$  is a non empty interval on  $\partial\Delta$ . For each  $i$ , let  $a_i$  be the center of  $[P_i, P_{i+1})$  and let  $\varepsilon = (1/3) \text{Min}_i |[P_i, P_{i+1})|$ , where  $|\cdot|$  denotes the length of an interval. We set

$$E_n = \{\gamma^{-1} a_{i_n}; \omega_\gamma = \gamma_{i_0} \dots \gamma_{i_n}, \gamma \in B_n - B_{n-1}\}.$$

For each  $\gamma$  in  $B_n - B_{n-1}$ ,  $\gamma$  maps  $Z(\omega_\gamma)$  onto  $[P_{i_n}, P_{i_n+1})$ , where  $\omega_\gamma = \gamma_{i_0} \dots \gamma_{i_n}$ . Therefore  $E_n$  is an  $(n, \varepsilon)$ -separated set. Hence  $t(n, \varepsilon) \geq \#E_n = M_n$  and the proof of Theorem 2 is completed.

*Remark.* Let  $\phi: \Gamma^g \rightarrow \text{PSL}(2, \mathbf{R})$  be a Fuchsian action. If a number  $m$  divides  $\chi(\phi) = \chi(\Sigma_g)$ , then  $\phi$  lifts to the  $m$ -sheeted covering group of  $\text{PSL}(2, \mathbf{R})$ . We denote the lifted homomorphism by  $\tilde{\phi}$ . Then  $\chi(\tilde{\phi}) = \chi(\phi)/m$ . On the other hand,  $h(\tilde{\phi}, \Gamma_0^g) = h(\phi, \Gamma_0^g)$  by Proposition 1. Therefore the converse of Theorem 2 does not hold.

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