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ON TREE-LIKE CONTINUA WHICH ARE HOMOGENEOUS WITH RESPECT TO CONFLUENT LIGHT MAPPINGS

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ABSTRACT. If X is a tree-like continuum with property K which is homogeneous with respect to confluent light mappings, then X contains no two nondegenerate subcontinua with the one-point intersection. This is a generalization of C. L. Hagopian's result concerning homogeneous X.

1. INTRODUCTION

A space X is homogeneous with respect to a class M of mappings if for every two points $x, y \in X$ there exists a mapping $f \in M$ of X onto X such that $f(x) = y$.

Many results concerning the generalized homogeneity have been obtained in recent years and a special interest was given to generalize theroems on homogeneous continua for some (larger than homeomorphisms) classes M of mappings (see [2]).

There are known one-dimensional continua which are homogeneous with respect to open light mappings but are not homogeneous (e.g. the one-point union of two Menger universal curves [2, p. 588]). However, we don't know such an example of a tree-like continuum. It was observed in [5] that if a continuum X is homogeneous with respect to open mappings and each proper subcontinuum of X is an arc, then X is not tree-like. Moreover, J. R. Prajs has recently proved that X is a solenoid, [8].

In this paper we present an immediate proof that no tree-like continuum with the property of Kelley which is homogeneous with respect to confluent light mappings contains two nondegenerate subcontinua with the one-point intersection. In particular, a tree-like continuum which is homogeneous with respect to open light mappings contains no two nondegenerate subcontinua with the one-point intersection. The theorem is a generalization of Hagopian's result about

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homogeneous X (see [3 and 6]). On the other hand, the theorem with some ideas of its proof have been used to prove that homogeneous tree-like continua are hereditarily indecomposable (see [7]).

2. Preliminaries

Throughout the paper all spaces are metric and all mappings are continuous. $C(X)$ denotes the hyperspace of all nonvoid subcontinua of a compact space X with the Hausdorff distance.

Let us recall that a mapping f of a compact space X onto Y is confluent if for every $A \in C(Y)$ and for every component B of $f^{-1}(A)$ we have $f(B) = A$. It is known that open mappings between compact spaces are confluent.

If a point $a \in X$ is fixed, we define the evaluation mapping $T_a : X^X \to X$ by $T_a(f) = f(a)$. Recall that the quasi-interior of $A \subset X$ is the set $A^* = \bigcup\{U : U$ is open in X and $U - A$ is of the first category$\}$. The following theorem holds [2, p. 584].

Proposition 2.1. *If a compact space X is homogeneous with respect to a Borel set $M \subset X^X$ and $a \in X$, then there is an $h \in M$ such that $h(X) = X$ and $h(a) \in T_a(H)^*$ for each H open in M with $h \in H$.*

A continuum A of a proper subcontinuum B of X is called an outlet subcontinuum of B if $A \subset Z$ for every subcontinuum Z of X such that $Z \cap B \neq \emptyset \neq Z - B$. If $A = \{a\}$, then a is called an outlet point of B [6].

The following propositions follow easily from the definitions.

Proposition 2.2. *If f is a confluent mapping of X and A is an outlet subcontinuum of $B \in C(X)$, then $f(A)$ is an outlet subcontinuum of $f(B)$.*

Proposition 2.3. *Suppose Y is a subcontinuum of a hereditarily unicoherent continuum X and $Y = K \cup L$, where K and L are subcontinua of X with distinct outlet point p and q, respectively, $p \in L$ and* $\operatorname{diam} L < \operatorname{diam} K$. *Then the subcontinuum of X which is irreducible between p and q is an outlet subcontinuum of Y.*

Let us recall that a continuum X has the property of Kelley (property K) if for every $y \in X$, for every sequence $\{y_n\} \subset X$ converging to y and for every $Y \in C(X)$ containing y there exists a sequence $\{Y_n\} \subset C(X)$ converging to Y such that $y_n \in Y_n$, $n = 1, 2, \dots$. Continua which are homogeneous with respect to open mappings have property K [1, p. 380] but homogeneity with respect to confluent mappings does not imply property K [4].

Proposition 2.4. *If a continuum X has property K and A_k is an outlet subcontinuum of $B_k \in C(X)$, $k = 1, 2, \dots$,* $\operatorname{Lim}_k A_k = A$, $\operatorname{Lim}_k B_k = B$, *then A is an outlet subcontinuum of B.*

A finite collection $\mathscr{C}$ of open subsets of X is called a tree-chain if its nerve is a tree. Elements of $\mathscr{C}$ are called links. A link $C \in \mathscr{C}$ is an end-link if

$\mathscr{C} - \{C\}$ is again a tree-chain; if C is not an end-link of $\mathscr{C}$, then C separates $\mathscr{C}$ and maximal tree-chains in $\mathscr{C} - \{C\}$ are called branches of $\mathscr{C} - \{C\}$. A continuum X is tree-like if there exists a sequence of tree-chains $\mathscr{C}_n$, $n = 1, 2, \dots$, covering X such that $\mathscr{C}_{n+1}$ refines $\mathscr{C}_n$ and $\lim_n \operatorname{mesh} \mathscr{C}_n = 0$. The sequence $\{\mathscr{C}_n\}$ will be called a defining sequence of X. By the diameter of a tree-chain $\mathscr{C}$ we understand the number $\operatorname{diam} \bigcup \mathscr{C}$.

3. Outlet points in tree-like continua

Lemma 3.1. *Let α be a positive number and let $\mathscr{D}$ be a tree-chain in a space X such that* $\operatorname{mesh}\mathscr{D} < \alpha/3$, $\alpha < \operatorname{diam}\bigcup\mathscr{D}$. *Denote by $\mathscr{D}^\circ$ the family of all branches of $\mathscr{D} - \{C\}$ having diameters $< \alpha/3$, where C runs separating links of $\mathscr{D}$. Let $\mathscr{D}'$ be the set of all such $C \in \mathscr{D}$ that there exists a branch of $\mathscr{D} - \{C\}$ (called associated with C) which is a maximal element of $\mathscr{D}^\circ$ with respect to inclusion. Fix an end-link $D \in \mathscr{D}$. Then there is $C \in \mathscr{D}'$ such that an associated branch of $\mathscr{D} - \{C\}$ does not contain D and exactly one branch $\mathscr{B}$ of $\mathscr{D} - \{C\}$ has the diameter $\geq \alpha/3$; moreover, $D \in \mathscr{B}$.*

Proof. Suppose the contrary. Take $C_1 \in \mathscr{D}'$, an associated branch of which does not contain D (clearly, it exists) and let $\mathscr{B}_1$ be a branch of $\mathscr{D} - \{C_1\}$ of diameter $\geq \alpha/3$ such that $D \notin \mathscr{B}_1$. Again, choose $C_2 \in \mathscr{D}' \cap \mathscr{B}_1$, an associated branch of which is in $\mathscr{B}_1$ (so, it does not contain D) and let $\mathscr{B}_2$ be a branch of $\mathscr{D} - \{C_2\}$ of diameter $\geq \alpha/3$ such that $D \notin \mathscr{B}_2$, etc. Continuing this process we get an infinite set $\{C_1, C_2, \dots\}$, a contradiction.

In the remaining part of this section, X will denote a nondegenerate tree-like continuum with defining sequence $\{\mathscr{C}_n\}$.

A subcontinuum Y of X is said to be a limit of branches if there are branches $\mathscr{D}_{n_k}$ of $\mathscr{C}_{n_k} - \{E_k\}$ for some separating links $E_k \in \mathscr{C}_{n_k}$, $n_1 < n_2 < \cdots$, such that $Y = \operatorname{Lim}_k \bigcup \mathscr{D}_{n_k}$.

Lemma 3.2. *There is a sequence $\{Y_k\}$, $k = 1, 2, \dots$, of nondegenerate subcontinua of X such that for each k*

(i) *Y_k is a limit of branches,*
(ii) *$Y_{k+1} \subset Y_k$,*
(iii) *$\bigcap_{k=1}^\infty Y_k$ is degenerate.*

Proof. Clearly, X contains a nondegenerate limit of branches Y_1. Suppose nondegenerate subcontinua $Y_1, Y_2, \dots, Y_{m-1} = Y$ of X satisfy (i) and (ii), $0 < \alpha < \operatorname{diam} Y/2$. To prove the lemma we shall construct a nondegenerate limit of branches $Y_m \subset Y$ with $\alpha/3 \leq \operatorname{diam} Y_m \leq \alpha$. Without loss of generality we can assume $Y = \operatorname{Lim}_n \bigcup \mathscr{D}_n$, where $\mathscr{D}_n$ is a branch of $\mathscr{C}_n - \{E_n\}$ for a separating link $E_n \in \mathscr{C}_n$ and mesh $\mathscr{D}_n < \alpha/3 < \alpha < \operatorname{diam} \bigcup \mathscr{D}_n$. Let D_n be an end-link of $\mathscr{D}_n$ disjoint with E_n. By Lemma 3.1, for each n there exists $C_n \in \mathscr{D}_n'$ such that an associated branch of $\mathscr{D}_n - \{C_n\}$ does not contain D_n and exactly one branch $\mathscr{B}_n$ of $\mathscr{D}_n - \{C_n\}$ has the diameter $\geq \alpha/3$; moreover, $D_n \in \mathscr{B}_n$. Then a link $B_n \in \mathscr{B}_n$ intersects C_n and the branch $\mathscr{A}_n$ of $\mathscr{D}_n - \{B_n\}$

containing all branches of $\mathscr{D}_n - \{C_n\}$ which are different from $\mathscr{B}_n$ is also a branch of $\mathscr{C}_n - \{B_n\}$ satisfying $\alpha/3 < \operatorname{diam} \bigcup \mathscr{A}_n < \alpha$. Finally, put $Y_m = \operatorname{Lim}_k \bigcup \mathscr{A}_{n_k}$ for a convergent subsequence $\operatorname{cl}(\bigcup \mathscr{A}_{n_k})$, $k = 1, 2, \ldots$.

Lemma 3.3 (See [6, Lemma 3.2]). *If X has property K and $Y \in C(X)$ is a limit of branches, then Y contains an outlet point.*

Lemmas 3.2 and 3.3 imply a stronger version of [6, Theorem 3.3].

Theorem 3.4. *If X has property K, then X contains a sequence of nondegenerate subcontinua Y_k such that for each $k = 1, 2, \ldots$*

(i) *Y_k has an outlet point,*

(ii) $Y_{k+1} \subset Y_k$,

(iii) *$\bigcap_{k=1}^{\infty} Y_k$ is degenerate.*

4. Main results

Theorem 4.1. *If a nondegenerate tree-like continuum X with property K is homogeneous with respect to confluent light mappings, then there exists a sequence $Z_k \in C(X)$ such that, for each k, Z_k contains a nondegenerate outlet subcontinuum,* $\operatorname{diam} Z_k < 1/k$ *and $\bigcap_{k=1}^{\infty} Z_k \neq \varnothing$.*

Proof. Theorem 3.4 guarantees the existence of a sequence $\{Y_k\}$ satisfying (i)–(iii). It suffices to consider the following two cases: (1) the only point p of $\bigcap_{k=1}^{\infty} Y_k$ is an outlet point of Y_k for each k; (2) for every such a sequence $\{Y_k\}$, each Y_k contains an outlet point $p_k \notin \bigcap_{k=1}^{\infty} Y_k$.

In case (1) let h be a confluent light mapping of X onto X which is described in Proposition 2.1 for the class M of all confluent light mappings and for $a = p$. Each $h(Y_k)$ is nondegenerate. By Proposition 2.2, $h(p)$ is an outlet point of $h(Y_k)$ for each k. Furthermore, $\{h(p)\} = \bigcap_{k=1}^{\infty} h(Y_k)$ and $h(Y_{k+1}) \subset h(Y_k)$. Fix a positive integer k. Let m be such that $\operatorname{diam} h(Y_m) < 1/4k$ and let $0 < \varepsilon < \operatorname{diam} h(Y_m)/2$. There exists a $\delta > 0$ such that $B(h(p), \delta) \subset T_p(H)^*$, where $B(h(p), \delta)$ denotes the δ-ball about $h(p)$ and H is the ε-ball about h in M. Let n be such that $n > m$, $h(Y_n) \subset B(h(p), \delta)$ and $\operatorname{diam} h(Y_n) < \operatorname{diam} h(Y_m) - 2\varepsilon$. Choose $q \in h(Y_n) - \{h(p)\}$. Since $q \in T_p(H)^*$ and $T_p(H)$ is dense in $T_p(H)^*$, there exists a sequence of points $q_s \in T_p(H)$, $s = 1, 2, \ldots$, converging to q. Hence, for each s there is $h_s \in H$ sending p onto q_s. Take a convergent subsequence $h_{s_i}(Y_m)$, $i = 1, 2, \ldots$, and denote its limit by K. In view of Proposition 2.2, q_{s_i} is an outlet point of $h_{s_i}(Y_m)$ and, by Proposition 2.4, q is an outlet point of K. Note that $\operatorname{diam} h(Y_n) < \operatorname{diam} K$. Define $Z_k = h(Y_n) \cup K$. Observe that $\operatorname{diam} Z_k \leq 2 \operatorname{diam} h(Y_m) < 1/2k$. We conclude by Proposition 2.3 that the unique subcontinuum of X irreducible between p and q is a nondegenerate outlet subcontinuum of Z_k. The above construction of Z_k shows that $h(p) \in Z_k$ for each k.

Assume now case (2) holds. Let $Y_k \in C(X)$ form a decreasing sequence with outlet points $p_k \in Y_k$, $\bigcap_{k=1}^{\infty} Y_k = \{p\}$, $p_k \neq p$ for each k. Fix a positive integer k. Take Y_n with $\operatorname{diam} Y_n < 1/2k$. There exists a confluent light mapping f of X onto X such that $f(p) = p_n$. Subcontinua $f(Y_m)$, $m = 1, 2, \ldots$, form again a sequence satisfying Theroem 3.4 and $\bigcap_{m=1}^{\infty} f(Y_m) = \{p_n\}$. Let m be such that $\operatorname{diam} f(Y_m) < \operatorname{diam} Y_n$. Assuming (2), $f(Y_m)$ contains an outlet point q_m different from p_n. In view of Proposition 2.3, the continuum $Z_k = Y_n \cup f(Y_m)$ contains a nondegenerate outlet subcontinuum. We have $\operatorname{diam} Z_k < 1/k$ and $p \in Z_k$ for each k.

So, the proof is complete.

The following theorem is an easy consequence of Theorem 4.1 and Proposition 2.2.

Theorem 4.2. *No tree-like continuum with property K which is homogeneous with respect to confluent light mappings contains two nondegenerate subcontinua with the one-point intersection.*

Corollary 4.3. *No tree-like continuum which is homogeneous with respect to open light mappings contains two nondegenerate subcontinua with the one-point intersection.*

Recall that $A \in C(X)$ is weakly terminal if there is a $\delta > 0$ such that if $Z \in C(X)$, $\operatorname{diam} Z < \delta$ and $Z \cap A \neq \emptyset$, then $Z \subset A$ [6]. It is easy to see (compare [6, Lemma 4.1]) that the continua Z_k in Theorem 4.1 are weakly terminal. So, we have the following generalization of [6, Theorem 4.2].

Corollary 4.4. *Each nondegenerate tree-like continuum with property K which is homogeneous with respect to confluent light mappings contains arbitrarily small, weakly terminal subcontinua with nondegenerate outlet subcontinua.*

If $A \subset X$ is an outlet subcontinuum of itself, then A is called a terminal subcontinuum of X.

Question 4.5. *Suppose X is a tree-like nondegenerate continuum with property K which is homogeneous with respect to confluent light mappings. Does X contain arbitrarily small, nondegenerate, terminal* (*indecomposable*) *subcontinua? Is X indecomposable?*

References

1. J. J. Charatonik, *The property of Kelley and confluent mappings*, Bull. Polish. Acad. Sci. Math. **31** (1983), 375–380.
2. J. J. Charatonik and T. Maćkowiak, *Around Effros' theorem*, Trans. Amer. Math. Soc. **298** (1986), pp. 579–602.
3. C. L. Hagopian, *No homogeneous tree-like continuum contains an arc*, Proc. Amer. Math. Soc. **88** (1983), pp. 560–564.
4. H. Kato, *Generalized homogeneity of continua and a question of J. J. Charatonik*, Houston J. Math. **13** (1987), 51–63.
5. P. Krupski *On continua each of whose proper subcontinua is an arc—a generalization of two theroems of R. H. Bing*, Colloq. Math. Soc. János Bolyai **41**, Topology and Applications, Eger (Hungary), 1983, pp. 367–371.

6. ———, *On homogeneous tree-like continua*, Rend. Circ. Mat. Palermo Serie II, **18** (1988), 327–336.

7. P. Krupski and J. R. Prajs, *Outlet points and homogeneous continua*, Trans. Amer. Math. Soc. (to appear).

8. J. R. Prajs, *Openly homogeneous continua having only arcs for proper subcontinua*, preprint.

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