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CHARACTERIZATIONS OF THE CONJUGACY OF SYLOW p-SUBGROUPS OF CC-GROUPS

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ABSTRACT. A group G is said to be a *CC-group* (*group with Černikov conjugacy classes*) if $G/C_G(x^G)$ is a Černikov group for all x in G. In this paper we show the characterization of the conjugacy of the Sylow p-subgroups of such a group G in terms of the number of them and of the inner structure of G.

1. INTRODUCTION

Groups with Černikov conjugacy classes or *CC-groups* were introduced by Polovickiĭ in [8] as a natural extension of the concept of FC-group, that is, a group in which every element has only a finite number of conjugates. A group G is said to be a CC-group if $G/C_G(x^G)$ is a Černikov group for all x in G. Polovickiĭ's Theorem of characterization of CC-groups assures that if G is a CC-group then x^G is Černikov-by-cyclic and $[G,x]$ is Černikov for every x in G (see [9, 4.36]).

By using the topological approach of [3] as a tool, a classical Sylow Theory of CC-groups was initiated in [1]. The main results obtained there can be recalled here as follows.

(1) *The Sylow p-subgroups of a CC-group are locally conjugate.*
(2) *The Sylow p-subgroups of a CC-group are conjugate if and only if they are countable in number.*

Here, if p is a prime, a Sylow p-subgroup of a group G is exactly a maximal p-subgroup of G. Thus, the above results extend well-known results of B. Neumann and M. I. Kargapolov for FC-groups (see [11, 5.2 and 5.8]). A very interesting fact in Kargapolov's study [4] of the conjugacy of the Sylow p-subgroups of an FC-group G is that such a property can be characterized in terms of the group G because the condition is equivalent to the Sylow p-subgroups of $G/O_p(G)$ being finite (see [11, 5.8]). This result can be easily deduced from the good behaviour of the conjugacy class of a subgroup of an FC-group with respect to the core of the subgroup and from the relationship

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of that class with the corresponding local conjugacy class (see [11, Chapter 4]). The corresponding results for CC-groups are reasonably far from being true or, at least, are unknown at the present moment. Nevertheless we can obtain a complete characterization of the conjugacy of the Sylow subgroups of a CC-group, which includes a new proof of the above (2). This is the main result of this paper that reads as follows.

Theorem. *Let p be a prime. For a CC-group G the following conditions are equivalent.*

(1) *The Sylow p-subgroups of G are conjugate in G.*
(2) *G has countably many Sylow p-subgroups.*
(3) *$G/O_p(G)$ satisfies* Min-p, *the minimal condition for p-subgroups.*
(4) *The Sylow p-subgroups of $G/O_p(G)$ are Černikov groups.*
(5) *The Sylow p-subgroups of $G/O_p(G)$ are finite groups.*
(6) *The torsion subgroup of G is a finite extension of a p'-extension of $O_p(G)$.*
(7) *There exists a normal subgroup N of G such that N is a p^*-extension of $O_p(G)$ and G/N is Černikov.*

It may be worthwhile noting that the condition (5) is identical to that of the FC-case and gives a strong finiteness condition on the structure of a G as in the theorem. For G arbitrary, the Sylow p-subgroups of $G/O_p(G)$ are FC-groups (see 2.2 below). Moreover, through the condition (5), we were able to give a new shorter proof of Theorem 2 of [1] and to avoid the use of the main result of [2]. In fact the proof given here of the step (2) $\Rightarrow$ (5) is due to Professor B. Hartley whom we wish to thank for allowing us to include it here. On the other hand, the conditions (6) and (7) are similar to those of [10]. As usual p' denotes the set of all primes different from p. We add the symbol p^* to be defined as $p^* = p' \cup \{\infty\}$. Thus a p^*-group G is a group with no elements of order p and a p'-group is a periodic p^*-group. These conditions are very close to (3) and all of them are related to a result attributed to Kargapolov: [5, 3.17]. In fact the latter can be used to show (6) but we have preferred to give an independent proof.

Throughout our group-theoretic notation is standard and is taken from [9]. If G is a group, we denote by $Z(G)$ the centre of G. We recall that, as in [9], if $x \in G$ we denote by x^G the normal closure of the subset $\{x\}$ in G. [11] and [5] are other good references in the framework of this paper. Finally we refer to [3] for the topological approach used in [1], which we will need to use in a concrete step of the proof of our theorem.

2. Proofs

We recall that in a CC-group G the set $T(G)$ of all elements in G with finite order is a locally normal and Černikov, characteristic subgroup of G: *the torsion subgroup of G* ([1, Lemma 1]). On the other hand it is easy to show that an $\mathfrak{F}$-perfect CC-group is nilpotent; see [6, 1.2] for example. Thus for a CC-group G the properties $\mathfrak{F}$-perfect, semi-radicable and radicable are identical ([9, Corollary to 9.23]) and therefore G contains a unique maximal radicable subgroup $R(G)$: *the radicable part of* G; the details of this can be found in [9, Part 2, p. 123]. The next result shows some properties of this subgroup.

2.1 **Lemma.** *Let R be the radicable part of a CC-group G.*

(1) *R' is central in G and G/R is an FC-group.*

(2) *$T(R)$ is central in R and FC-central in G.*

Proof. (1) Let $x \in G$. $R/C_R(x^G)$ is Černikov and radicable so that it must be abelian and so $R' \leq C_R(x^G)$. Thus $R' \leq Z(G)$. On the other hand, if D is the radicable part of $[G,x]$ then $D \leq R$ and $[G,x]/D$ is finite. Therefore $[G,x]R/R$ is finite and $x^G R/R$ is finite-by-cyclic. By [9, Corollary 3 to 4.32] G/R is an FC-group.

(2) Let $x \in T(R)$. x^G is Černikov and nilpotent because one has that $x^G \leq R$. By [9, Corollary to 3.29.2] $G/C_G(x^G)$ is finite and so $R/C_R(x^G)$ must be trivial. Therefore x is FC-central in G and central in R.

As a consequence of 2.1 we obtain that a periodic radicable subgroup of a CC-group is Abelian. Other consequences are the following.

2.2 **Lemma.** *Let G be a CC-group and let p be a prime.*

(1) *$O_p(G)$ contains any radicable p-subgroup of G.*

(2) *If P is a Sylow p-subgroup of G then $P/O_p(G)$ is an FC-group. Moreover if $P/O_p(G)$ is Černikov then it is finite.*

Proof. (1) If Q is a radicable p-subgroup of G then $Q \leq R$, where R is the radicable part of G. If $x \in Q$ then $x^G \leq R$ and so x^G is nilpotent by (2.1). Therefore x^G is a p-group and then $x^G \leq O_p(G)$. Hence $Q \leq O_p(G)$.

(2) By (1) $R(P) \leq O_p(G)$ and by 2.1 $P/R(P)$ is an FC-group. If $P/O_p(G)$ is Černikov then its radicable part is trivial by (1) and so $P/O_p(G)$ is finite.

The next result will allow us to reduce some aspects of the proof of the Theorem modulo the centre.

2.3 **Lemma.** *Let G be a periodic CC-group and put $Z = Z(G)$, $L = G/Z$. If p is a prime then $O_p(L) = O_p(G)Z/Z$ and the Sylow p-subgroups of $G/O_p(G)$ and of $L/O_p(L)$ are isomorphic.*

Proof. Let $H/Z = O_p(L)$. It is clear that H is locally nilpotent so that $O_p(H)$ is its unique Sylow p-subgroup. Since $O_p(G) \leq H$ we have that $O_p(G) = O_p(H)$. By [1, Corollary to Theorem 1] $O_p(G)Z/Z$ is a Sylow p-subgroup of $O_p(L)$ and then we find the equality.

Now let P be a Sylow p-subgroup of G. By the above results $PZ/O_p(G)Z$ is isomorphic to a Sylow p-subgroup of $L/O_p(L)$. Clearly $PZ/O_p(G)Z$ is isomorphic to $P/O_p(G)$ since $P \cap Z \leq O_p(G)$. Then the result follows.

Proof of the theorem. Clearly $(3) \Leftrightarrow (4)$. Obviously $(5) \Rightarrow (4)$ and, by 2.2, $(4) \Rightarrow (5)$, $(3) \Rightarrow (2)$ and $(3) \Rightarrow (1)$ follow from [1, Lemma 3].

In what follows we shall denote by T the torsion subgroup of G. If $G = T$ we recall that the Sylow p-subgroups of G are well behaved with respect to normal subgroups and quotients by [1, Corollary to Theorem 1], which will be implicitly used in the sequel.

$(7) \Rightarrow (6)$. Assume that $O_p(G) = 1$ and let N be a normal p^*-subgroup of G such that G/N is a Černikov group. Let P be a Sylow p-subgroup of G. Clearly $P \cap N = 1$ and so P is isomorphic to a subgroup of G/N. Therefore P is Černikov and so finite by 2.2. On the other hand $T \cap N$ is a p'-group and $T/T \cap N$ is a Černikov group whose Sylow p-subgroups are finite groups. If $H/T \cap N$ is the radicable part of $T/T \cap N$ then $H/T \cap N$ is a direct product of finitely many Prüfer groups and its Sylow p-subgroups are again finite groups. Therefore $H/T \cap N$ is a p'-group and so is H. Thus (6) follows.

$(6) \Rightarrow (5)$. Again we assume that $O_p(G) = 1$. By hypothesis $T/O_{p'}(G)$ is finite. Thus a Sylow p-subgroup of G is finite because it is isomorphic to a subgroup of $T/O_{p'}(G)$.

$(5) \Rightarrow (7)$. Another time we may suppose that $O_p(G) = 1$. If P is a Sylow p-subgroup of G then P is finite and so P^G is Černikov. Since $G/Z(G)$ is residually Černikov and $P^G Z(G)/Z(G)$ satisfies Min, there exists a normal subgroup N of G such that $P^G \cap N \leq Z(G) \leq N$ and G/N Černikov. Since $O_p(G) = 1$ it is clear that $Z(G)$ is a p^*-group and so $P^G \cap N$ is a p^*-group. It is easy to show that G/P^G is a p^*-group and then $N/P^G \cap N$ is a p^*-group. Therefore N is a p^*-group.

$(2) \Rightarrow (5)$. Assume that the Sylow p-subgroups of $G/O_p(G)$ are infinite but G has only countably many Sylow p-subgroups and obtain a contradiction. We may assume that G is periodic and $O_p(G) = 1$. Clearly the map $P \to PZ(G)/Z(G)$ is a bijection between the Sylow p-subgroups of G and those of $G/Z(G)$. By 2.3 we may replace G by $G/\overline{Z}(G)$, where $\overline{Z}(G)$ is the hypercentre of G, and then assume $Z(G) = 1$.

We claim that we may construct recursively finite subgroups $F_1, \dots, F_n$ such that $\langle F_1, \dots, F_n \rangle = F_1 \times \cdots \times F_n$ and F_i has more than one Sylow p-subgroup. Suppose we have $F_1, \dots, F_n$ and let $F = \langle F_1, \dots, F_n \rangle$. Here G is residually Černikov and F^G is Černikov so that there exists a normal subgroup N of G such that G/N is Černikov and $F^G \cap N = 1$. Thus $N \leq C_G(F)$ and $F \cap N = 1$. Let P be an infinite p-subgroup of G. Then $P \cap N \neq 1$, otherwise P is Černikov and hence finite by 2.2. Therefore N contains a nontrivial finite p-subgroup Q. Since $O_p(N) = 1$ it follows that Q^N is not a p-group so there exist $x_1, \dots, x_r$ in N such that $\langle Q^{x_1}, \dots, Q^{x_r} \rangle$ is not a p-subgroup. Put $F_{n+1} = \langle Q, x_1, \dots, x_r \rangle$. Thus our claim has been carried out and then $F_1 \times \cdots \times F_n \times \cdots \leq G$ so that G has an uncountable number of Sylow p-subgroups.

$(1) \Rightarrow (5)$. As in [1, pp. 511–512] it can be shown that the Sylow p-subgroups of G are in fact conjugate in T. In order to show (5), we may replace G by $T/O_p(G)$ to assume that G is periodic and that $O_p(G) = 1$.

It suffices to show that a countable p-subgroup Q of G is finite. Since G is a periodic CC-group, the normal closure H of Q in G can be generated by Černikov groups and, in particular, H is countable. Here H is normal in G and $O_p(H) = 1$ so, by 2.3, if $\overline{Z}(H)$ is the hypercentre of H, we may replace H by $H/\overline{Z}(H)$ and G by $G/\overline{Z}(H)$ to assume $Z(H) = 1$. Moreover, by [7, Theorem 6], H can be viewed as a subgroup of the direct product D of a countable family $\{C_n | n \geq 1\}$ of Černikov groups.

Fix a Sylow p-subgroup P of H. Since H is normal in G we note that the Sylow p-subgroups of H have the form P^x where x runs through G. Let J be a finite subset of $\mathbb{N}$ and put $K_J = Dr\{C_n | n \in J\}$. K_J is Černikov, $H \cap K_J$ is normal in H and $P \cap K_J$ is a Sylow p-subgroup of $H \cap K_J$. We claim that there is an integer $m \geq 1$ such that if J is any finite subset of $\mathbb{N} - \{1, \dots, m\}$ then $P \cap K_J$ is normal in H. For, suppose that our claim is false. Starting from the case $m = 1$, there is a finite subset F of $\mathbb{N} - \{1\}$ such that $P \cap K_F$ is not normal in H. Let n_1 be the maximum of F so that $F \subseteq J_1 = \{1, \dots, n_1\}$. Clearly $P \cap K_{J_1}$ cannot be normal in H so that we may choose $F = J_1$. By applying to $\mathbb{N} - J_1$ the same argument and proceeding inductively we find a partition $\mathcal{J} = \{J_i | i \geq 1\}$ of $\mathbb{N}$ into finite sets such that $P \cap K_{J_i}$ is not normal in H for every $i \geq 1$. Put K_i instead of K_{J_i} and choose x_i in H such that $P^{x_i} \cap K_i = (P \cap K_i)^{x_i} \neq P \cap K_i$. If i and j are different indexes it is clear that every element of $P^{x_i} \cap K_i$ commutes with every element of $P^{x_j} \cap K_j$. Therefore the subgroup S of H given by $S = \langle P^{x_i} \cap K_i | i \geq 1\rangle$ is a p-group so that $S \leq P^x$ for some x in G.

Let $H = \{h_1, \dots, h_n, \dots\}$, $H_i = \langle h_1, \dots, h_i\rangle^G$, $B_i = C_G(H_i)$ and $C = C_G(H)$. G/C is then a co-Černikov group with respect to $\{B_i/C | i \geq 1\}$ (see [3]). Any centralizer in a co-Černikov group is always closed so, by [3, 2.13], there is some index t such that $B_t/C \leq C_{G/C}(x^G C/C)$. From this, it follows that $C_H(H_t)$ is contained in $C_H(x^G)$ since $Z(H) = 1$. Now, for each $i \geq 1$, we define $D_i = Dr\{K_j | j > i\}$. Clearly we have a descending chain $D > D_1 > \cdots > D_i > \cdots$ of normal subgroups of D such that D/D_i is Černikov and $\cap\{D_i | i \geq 1\} = 1$ and we think of D as co-Černikov group with respect to $\mathcal{D} = \{D_i | i \geq 1\}$. H is now a co-Černikov group with respect to $H \cap \mathcal{D}$ and, as above, there is some $r \geq 1$ such that $H \cap D_r \leq C_H(H_t)$. Thus $S \cap D_r \leq C_S(H_t) \leq C_S(x^G)$. It is clear that $S \cap D_i = \langle P^{x_j} \cap K_j | j > i\rangle$ and so, for every $j > r$, $P^{x_j} \cap K_j \leq C_S(x^G)$. Since $S \leq P^x$, we have that $C_S(x^G) \leq P^x \cap P$ and then $P^{x_j} \cap K_j = P \cap K_j$, a contradiction which shows our claim. Now let $K = Dr\{C_n | n > m\}$. By the election of m it follows that $P \cap K$ is normal in H and therefore $P \cap K = 1$ since $O_p(H) = 1$. Thus P is isomorphic to a subgroup of D/K and so P is Černikov. By 2.2 P is finite and then the proof is now complete.

It is an open question to decide whether one can find a subgroup N as in (7) of finite index in G. The key point to do that seems to be the knowledge of the Sylow p-subgroups of the quotient G/N. It is clear that if G/N is a Černikov group and the Sylow p-subgroups of G/N are finite groups then the Sylow p-subgroups of the radicable part H/N of G/N are trivial. Therefore, in the situation of (7), H would be a p^*-extension of $O_p(G)$ and the question would be solved.

Added in proof. M. Tomkinson has pointed out that the answer to the last question is affirmative. For, let G be as in the theorem with $O_p(G) = 1$ and suppose that N is a normal p^*-subgroup of G such that G/N is Černikov. If R/N is the radicable part of G/N, then $R' \leq N$ and so R' is a p'-group. Thus the Sylow p-subgroup P/R' of R/R' is finite (by (5)) and pure, so it is a direct factor of R/R'. If $R/R' = (P/R') \times (Q/R')$, then Q is a p^*-group and R/Q is finite. Hence G is a p^*-group-by-finite.

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