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WEAKLY CONSTRICTED OPERATORS AND JAMISON'S CONVERGENCE THEOREM

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ABSTRACT. If T is a linear contraction on a complex $C(X)$ space which is irreducible and if $\{T^n f\}$ converges weakly to zero then the convergence is actually in norm. If T is weakly constricted and irreducible then T is actually strongly constricted. If T is weakly almost periodic and irreducible and either T or T^* has a unimodular point spectral value then T is strongly almost periodic.

1. AN EXTENSION OF JAMISON'S CONVERGENCE THEOREM

We consider a bounded linear operator T on the space of complex-valued continuous functions on a compact Hausdorff space X with $\|T\| \leq 1$. For each point p in X then $\mu = T^*\delta(p)$ is a complex Radon measure on X with total variation norm of at most 1. The *support* of such a measure is the minimal closed set, $\operatorname{supp}(\mu)$, so that for G open and disjoint from $\operatorname{supp}(\mu)$ we have $\mu(G) = 0$.

A closed subset E in X is *invariant* for T if for each point p in E we have $\operatorname{supp}(T^*\delta(p))$ contained in E. We call T *irreducible* if the only invariant sets for T are X and $\varnothing$.

The first theorem here extends Jamison's convergence theorem [J] from an irreducible $C(X)$ Markov operator to an irreducible contraction on $C(X)$. Most of Jamison's proof remains valid for this setting. A slight extra effort is needed as the measures $\{T^*\delta(p)\}$ are no longer probabilities. We quote Jamison's first lemma for completeness with the format and notation adopted from the exposition in Krengel [Kr, p. 180].

Lemma 1. *Let X be a compact metric space. Suppose $\{f_n\}$ is a uniformly bounded sequence of members of $C(X)$. Define*

$$u_{m,n}(x) = \sup\left\{|f_n(y)| : d(x,y) \leq \frac{1}{m}\right\},$$

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and

$$u_m(x) = \limsup_n u_{m,n}(x),$$

and finally $u(x) = \lim u_m(x)$. *(The last limit exists as* $u_m(x)$ *is pointwise monotone decreasing.) Then the following hold*

(A) u *is upper semicontinuous and so assumes on* X *its supremum* α,
(B) $\|(f_n| \vee \alpha) - \alpha\| \to 0$,
(C) *if, in addition,* $f_n \to 0$ *[weakly], then the set* $\{x: u(x) > 0\}$ *is first category.*

Remark. Here, of course, $|z|$ is the modulus of the complex value z rather than the absolute value of a real value as in Jamison's original lemma. A moment's reflecion shows that the standard analysis arguments of the original lemma are still valid.

Theorem 2. *Let* T *be an irreducible contraction on* $C(X)$. *If* $T^n f \to 0$ *[weakly] then* $\|T^n f\|$ *converges to zero.*

Proof. We assume first that X is compact metric and we set $f_n = T^n f$. Again let α be $\sup\{u(x): x \in X\}$. If $\alpha = 0$ then, by (B) of the lemma, $\|T^n f\|$ converges to zero. So we assume $\alpha > 0$ and set $E = \{x: u(x) = \alpha\}$. As u is upper semicontinuous, the set E is nonempty and closed. The next part of the argument is to show that E is invariant. For then $E = X$ by irreducibility while (C) would imply that X is of first category. This contradiction will yield the desired strong convergence then in the case that X is a compact metric space.

To show E is irreducible we assume that there is a point p in E with $\operatorname{supp}(T^*\delta(p))$ not contained in E. Then there is a closed set D disjoint from E so that $|\mu(D)| > 0$ where $\mu = T^*\delta(p)$.

As p is in E we have $u(p) = \alpha$ which implies there is a sequence of points $\{p_k\}$ in E and a strictly increasing sequence of integers $\{n_k\}$ so that

$$p_k \to p$$

and

$$|T^{n_k+1} f(p_k)| = |f_{n_k+1}(p_k)| \to \alpha.$$

We let σ_k denote the total variation measure of $\mu_k = T^*\delta(p_k)$ and let σ_0 be any weak-$*$ cluster point of $\{\sigma_k\}$. Then set $\sigma' = \sigma_0 + \sigma$.

For any fixed $r > 0$, we define the open set

$$G(r) = \bigcup\{B(q;r): q \in D\}$$

where $B(q;r)$ is the open ball about q of radius r. Now

$$\bigcap\{G(r): r > 0\} = D$$

so there is a range $0 < r \le r_0 < \operatorname{dist}(D, E)$ for which $\mu(G(r))$ is nonzero. Note that $\sigma'(G(r))$ is a positive monotone function of r in this range and so has but a countable number of jumps. We then can take r^* to be a continuity point of $\sigma'(G(r))$ with $0 < r^* < r_0$. Then the set F, defined as the closure of $G(r^*)$, is disjoint from E and $\sigma'(\partial F) = 0$. Thus $\mu_k(F) \to \mu(F)$ as $\sigma(\partial F) = 0$. Also, since $\sigma_0(\partial F) = 0$, we can write $|\mu_k(F)| \le \sigma_k(F) \to \sigma_0(F)$. Hence $\sigma_0(F)$ is strictly positive. Moreover $\sigma_k(F) \to \sigma_0(F)$ and $\sigma_k(F^c) \to \sigma_0(F^c)$ over some subsequence.

We know F is closed and disjoint from E so upper semicontinuity of u implies that $\beta = \sup\{u(x) : x \in F\}$ is less than α. We set $\varepsilon = (1/2)(\alpha - \beta)$. Now write

$$T^{n_k+1} f(p_k) = \int_F T^{n_k} f \, d\mu_k + \int_{F^c} T^{n_k} f \, d\mu_k.$$

Then

$$|T^{n_k+1} f(p_k)| \le \int_F |f_{n_k}| \, d\sigma_k + \int_{F^c} |f_{n_k}| \, d\sigma_k.$$

We know from (B) that if $|f_{n_k}| \vee \alpha \to \alpha$ [uniformly] on all of X so certainly on F^c. Suppose that

$$|f_{n_k}| \vee (\alpha - \varepsilon) \to \alpha - \varepsilon \quad \text{[uniformly]}$$

on F. Then, because $\sigma_0(F) > 0$ and $\sigma_0(F) + \sigma_0(F^c) \le 1$, we obtain

$$\alpha \le (\alpha - \varepsilon)\sigma_0(F) + \alpha\sigma_0(F^c) < \alpha.$$

This contradiction shows that the assumed uniform convergence on F does not hold. Thus we can find a subsequence $\{m_j\}$ of $\{n_k\}$ and points $\{q_j\}$ in F with

$$|T^{m_j} f(q_j)| > \alpha - \varepsilon.$$

Without loss of generality we can assume that $\{q_j\}$ converges to a point q_0 which will necessarily be in F. But then

$$u(q_0) \ge \limsup |f_{m_j}(q_j)| \ge \alpha - \varepsilon > \alpha - 2\varepsilon = \beta.$$

This contradicts the definition of β so proves the strong convergence of $\{T^n f\}$ under the assumption that X is metric.

To establish the result for an arbitrary compact Hausdorff space observe that f is contained in a separable closed unitary self-adjoint T invariant subalgebra of $C(X)$. We then apply the metric result to $\widehat{T}$ on $C(\widehat{X})$ where $\widehat{X}$ is the resulting compact metric identification space. This finishes the proof. □

Remark. Jamison used his convergence theorem to show that an irreducible weakly almost periodic Markov operator is strongly almost periodic. Also needed for this is an observation of Rosenblatt [R] that, under the assumptions, $\{T^n f\}$ clustering weakly to zero implies weak convergence to zero. (Also see [Kr, p. 182].) This last observation exploits the existence of an invariant probability measure. In our situation there may be no nontrivial invariant functionals. We consider what can be done next.

2. Almost periodic operators

A contraction T on a Banach space $\mathscr{X}$ is *strongly almost periodic* if for each $x \in \mathscr{X}$ any subsequence of $\{T^n x\}$ admits a further subsequence which is strongly convergent. If each subsequence of $\{T^n x\}$ admits a further subsequence which is weakly convergent we call T *weakly almost periodic.* We refer the reader to Krengel [Kr, p. 103] for a discussion of the properties of such operators and the related deLeeuw-Glicksberg decomposition.

Theorem 3. *If T is an irreducible weakly almost periodic contraction on $C(X)$ and either T or T^* has a unimodular point spectral value then T is strongly almost periodic.*

Proof. Observe that if $|\lambda| = 1$ then $R = (T/\lambda)$ is weakly almost periodic if and only if T is. If 1 is in the point spectrum of R^* then $R^*\mu = \mu$ for some nonzero functional μ. Pick f in $C(X)$ so that $(\mu, f) \neq 0$. Then

$$\frac{1}{N+1}\sum_0^N R^n f$$

must converge, as weak almost periodicity implies mean ergodicity, to an invariant function g. Moreover $(\mu, g) = (\mu, f)$ so g is nonzero.

It is easy to see that the closed subset of X where g achieves its maximum modulus is invariant. As X is assumed irreducible this implies that g is of constant (nonzero) modulus. Then

$$S = g^{-1} R g$$

is a contraction with $S1 = 1$. This means that S is a Markov operator on $C(X)$. Clearly S is irreducible and weakly almost periodic. Thus S, and therefore T as well, is strongly almost periodic. □

Remark. The obvious open question is whether or not irreducibility and weak almost periodicity alone suffice for strong almost periodicity. The possible absence of any nontrivial invariant structure is what apparently makes this hard. We will see how to get around this difficulty for a constricted operator in the next section.

3. Constricted systems

A contraction T on a Banach space $\mathscr{X}$ is *strongly constricted* if there is a set K in $\mathscr{X}$ which is strongly compact and has the property that for all x in $\mathscr{X}$ with $\|x\| \leq 1$ we have

$$\operatorname{dist}(T^n x, K) \to 0.$$

The operator is *weakly constricted* if this norm attractor property holds for a set K which is weakly compact. Almost the entire book of Lasota and Mackey [LM] is devoted to constricted systems and their applications. Komornik [Ko] proved that a weakly constricted L_1 Markov operator is strongly constricted. In [S] it is shown that a weakly constricted $C(X)$ Markov operator need not be strongly constricted. We show next that irreducibility makes all the difference.

Corollary 4. *Let T be an irreducible weakly constricted contraction on $C(X)$. Then T is strongly constricted.*

Proof. It is clear that T is weakly almost periodic. The deLeeuw–Glicksberg decomposition yields

$$C(X) = \mathscr{X}_0 \oplus \mathscr{X}_1$$

where $\mathscr{X}_1$ is the closed span of the unimodular eigenfunction and each x in $\mathscr{X}_0$ has the property that zero is a weak cluster point of $\{T^n x\}$. It was shown in [S, Theorem 7] that weak convergence to zero actually holds for x in $\mathscr{X}_0$. Thus the extended version of Jamison's convergence theorem can be applied to obtain norm convergence. It is also known from [S] that $\mathscr{X}_1$ is finite dimensional. Thus T is strongly constricted with the unit ball of $\mathscr{X}_1$ serving as a norm compact attractor K. □

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