

## ISOMETRIES OF SPACES OF WEAK \* CONTINUOUS FUNCTIONS

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**ABSTRACT.** If  $X$  is a compact Hausdorff space and  $E^*$  a Banach dual, we denote by  $C(X, (E^*, \sigma^*))$  the Banach space of continuous functions from  $X$  to  $E^*$  when the latter space is given its weak \* topology. It is shown that if  $E_1^*$ ,  $E_2^*$  have trivial centralizers and satisfy a topological condition introduced by Namioka and Phelps, then, given an isometry  $T$  mapping  $C(X_1^*, (E_1^*, \sigma^*))$  onto  $C(X_2^*, (E_2^*, \sigma^*))$  and given  $F \in C(X_1, (E_1^*, \sigma^*))$ , there exists a dense  $G_\delta$  in  $X_2$  on which

$$(*) \quad (TF)(x) = U(x)F \circ \Phi(x),$$

where  $\Phi$  is a homeomorphism of  $X_2$  onto  $X_1$  and  $U(\cdot)$  an operator-valued function independent of  $F$ . If the  $E_i^*$  are separable then the  $U(x)$  are surjective isometries and  $x \rightarrow U(x)$  is continuous. When the  $X_i$  are metric the representation  $(*)$  holds on all of  $X_2$ .

### 1. INTRODUCTION

Various authors, beginning with M. Jerison [6], have considered the problem of determining geometric conditions on a Banach space  $E$  which would allow generalizations of the classical Banach-Stone theorem to spaces of the form  $C(X, E)$  consisting of norm-continuous vector-valued functions from the compact Hausdorff space  $X$  to  $E$ . The most comprehensive account of the history of the problem can be found in the monograph by E. Behrends [1], which contains Behrends' own results concerning Banach spaces  $E$  with trivial centralizer  $Z(E)$ . (For the definition and properties of the centralizer of a Banach space we refer the reader to [1].)

More recently, analogous questions have been asked concerning isometries of Banach spaces of the form  $C(X, (E^*, \sigma^*))$ , consisting of continuous functions from  $X$  to a Banach dual  $E^*$  when the latter space is given its weak \* topology, again normed by  $\|F\|_\infty = \sup_{x \in X} \|F(x)\|$ . However the only article known to the authors which treats the descriptive aspects of such mappings is [3] where positive results were obtained with regard to isometries  $T : C(X_1, (E_1^*, \sigma^*)) \rightarrow C(X_2, (E_2^*, \sigma^*))$  for hyperstonean  $X_i$  and Banach duals  $E_i^*$  with RNP and  $Z(E_i^*) = \mathbf{K}$ . (Here, and throughout this article,  $\mathbf{K}$  denotes the scalar field under consideration:  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{C}$ .) The object of this paper is to show that,

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for a class of Banach duals  $E_i^*$ , one can remove the assumption concerning the hyperstonean nature of the  $X_i$ , and that if the  $X_i$  are metric considerably more may be established.

The condition that we shall frequently impose on Banach duals in this article is that they satisfy the property (\*\*) considered by Namioka [8, p. 530], and by Namioka and Phelps [9, p. 741], namely that the weak\* and norm topologies coincide on the surface of the unit ball. We note that the class of spaces satisfying this property contains all locally uniformly convex Banach duals, as well as all spaces of the form  $\mathcal{L}^1(\Gamma)$  considered as the dual of  $c_0(\Gamma)$ , ( $\Gamma$  discrete), [9].

We remark that, for those who prefer the terminology and flavor of operator theory, all of our results may be formulated in terms of spaces of  $C(X)$ -valued operators, as  $C(X, (E^*, \sigma^*))$  is isometric to  $\mathcal{B}(E, C(X))$  [4, p. 490].

## 2. THE ISOMETRIES OF $C(X, (E^*, \sigma^*))$ SPACES

Following the terminology of [3] we shall say that an isometry  $R : C(X_1, (E^*, \sigma^*)) \rightarrow C(X_2, (E^*, \sigma^*))$  is "induced" by a homeomorphism  $\Phi$  from  $X_2$  onto  $X_1$  if  $RF = F \circ \Phi$  for all  $F \in C(X_1, (E^*, \sigma^*))$ . And an isometry  $S : C(X, (E_1^*, \sigma^*)) \rightarrow C(X, (E_2^*, \sigma^*))$  will be called a  $C(X)$ -module isomorphism if  $S(fF) = f \cdot SF$  for all  $f \in C(X)$  and  $F \in C(X, (E_1^*, \sigma^*))$ .

The following theorem and its proof parallel those [3, Theorem 2.2]. The only additional fact the proof here requires is provided by [2, Theorem 2.4].

**Theorem 1.** *Let  $X_i$  be compact Hausdorff spaces and  $E_i^*$  Banach duals with  $Z(E_i^*) = \mathbf{K}$ ,  $i = 1, 2$ . If  $T : C(X_1, (E_1^*, \sigma^*)) \rightarrow C(X_2, (E_2^*, \sigma^*))$  is a surjective isometry then  $X_1$  and  $X_2$  are homeomorphic. Furthermore,  $T$  admits a decomposition  $T = S \circ R$  into surjective isometries  $R : C(X_1, (E_1^*, \sigma^*)) \rightarrow C(X_2, (E_1^*, \sigma^*))$  and  $S : C(X_2, (E_1^*, \sigma^*)) \rightarrow C(X_2, (E_2^*, \sigma^*))$  where  $R$  is induced by a homeomorphism  $\Phi : X_2 \rightarrow X_1$  and  $S$  is a  $C(X_2)$ -module isomorphism.*

**Theorem 2.** *Let  $X_i$  be compact Hausdorff spaces and let  $E_i^*$  be Banach duals with  $Z(E_i^*) = \mathbf{K}$  for  $i = 1, 2$ , and such that  $E_1^*$  satisfies property (\*\*). Let  $T : C(X_1, (E_1^*, \sigma^*)) \rightarrow C(X_2, (E_2^*, \sigma^*))$  be a surjective isometry. Then there exists a homeomorphism  $\Phi$  of  $X_2$  onto  $X_1$  and there exist operators  $U(x) : E_1^* \rightarrow E_2^*$  with  $\|U(x)\| \leq 1$  for  $x \in X_2$  such that for every  $F \in C(X_1, (E_1^*, \sigma^*))$  we have*

$$(1) \quad (TF)(x) = U(x)(F \circ \Phi)(x)$$

for all  $x$  belonging to a set  $G_F$  which is a dense  $G_\delta$  in  $X_2$ .

*Proof.* Because of Theorem 1, if we seek representations for  $T$ , it suffices to find representations for  $C(X_2)$ -module isomorphisms  $S$  and then replace  $F \in C(X_2, (E_1^*, \sigma^*))$  by  $F \circ \Phi$ , where  $\Phi$  is the homeomorphism of that theorem. Hence we may assume, without loss of generality, that  $X_1 = X_2 = X$  and that  $T$  is itself a  $C(X)$ -module isomorphism.

Thus for each  $x \in X$  we define a linear operator  $U(x) : E_1^* \rightarrow E_2^*$  by  $U(x)e^* = (T(e^*))(x)$ , for  $e^* \in E^*$ . (Throughout this article, constant functions are denoted by boldface type.) Clearly  $\|U(x)e^*\| \leq \|e^*\|$  for all  $x \in X$ .

We first claim that if  $x$  is a point of  $X$  at which  $F$  is norm continuous, then  $(TF)(x) = T(\mathbf{F}(\mathbf{x}))(x) (= U(x)\mathbf{F}(\mathbf{x}))$ . For this we show that if  $x \in X$  is such that  $F$  is norm continuous at  $x$  and  $F(x) = 0$  then  $(TF)(x) = 0$ . Thus fix such a point  $x$  and let  $\{U_i : i \in I\}$  be the family of neighborhoods of  $x$ , where the set of indices  $I$  is directed in the usual manner by set inclusion ( $i_1 \leq i_2$  if  $U_{i_2} \subseteq U_{i_1}$ ). For each index  $i$  let  $f_i$  be a continuous scalar function,  $f_i : X \rightarrow [0, 1]$ , such that  $f_i(x) = 1$  for all  $i$  and the support of  $f_i$  is contained in  $U_i$ . Then since  $T$  is a  $C(X)$ -module isomorphism we have  $(TF)(x) = f_i(x)(TF)(x) = (Tf_i F)(x)$ . Since  $\|f_i F\|_\infty \rightarrow 0$  we have established our claim.

Thus, to end the proof, we show that if  $F$  is any element of  $C(X, (E_1^*, \sigma^*))$ , then, on a dense  $G_\delta$  subset of  $X$ ,  $F$  is continuous from  $X$  to  $E_1^*$ , when  $E_1^*$  is given its norm topology. By passing from  $F$  to  $F$  plus a large constant if necessary, we may assume without loss of generality that  $F(x) \neq 0$  for any  $x$ . By [5, p. 87] there exists a set  $G_F$  which is a countable intersection of dense open subsets of  $X$  on which  $\|F\|$  is continuous. Hence  $F/\|F\|$  is weak\* continuous on  $G_F$  and, since the latter function has norm one and takes its values in a space satisfying property (\*\*), it is actually norm continuous on  $G_F$ . Thus  $F = \|F\|(F/\|F\|)$  is norm continuous on  $G_F$  so we are done.

**Theorem 3.** *If, in addition to the assumptions of Theorem 2, the  $E_i^*$  are separable,  $i = 1, 2$ , then for all  $x$  belonging to a subset  $G$  which is a dense  $G_\delta$  in  $X_2$ ,  $U(x)$  is a surjective isometry. Hence, in particular,  $E_1^*$  and  $E_2^*$  are isometric. Moreover, at every point  $x \in G$ ,  $U(\cdot)$  is continuous as a map from  $X_2$  to  $\mathcal{B}(E_1^*, E_2^*)$  when the latter space is given its strong operator topology.*

*Proof.* Again we may assume that  $X_1 = X_2 = X$  and that  $T$  is a  $C(X)$ -module isomorphism. Recall that, for  $x \in X$ ,  $U(x)$  is defined by  $U(x)e^* = (T(e^*))(x)$ ,  $e^* \in E_1^*$ . Then for fixed  $e^* \in E_1^*$  we claim that, on a countable intersection of open dense subsets of  $X$  we have  $\|U(x)e^*\| = \|e^*\|$ .

For it follows from the weak\* continuity of  $T(e^*)$  that the function  $\|U(x)e^*\|$  is lower semicontinuous on  $X$ . Thus for each  $n = 1, 2, \dots$  the set  $A_n := \{x \in X : \|U(x)e^*\| \leq \|e^*\| - 1/n\}$  is a closed subset of  $X$ . If we assumed that, for some  $n$ ,  $A_n$  had nonvoid interior, then we could find an  $f \in C(X)$ ,  $f : X \rightarrow [0, 1]$ , such that the support of  $f$  is contained in  $A_n$  and  $f(x) = 1$  for some  $x \in A_n$ . We would thus have  $\|fT(e^*)\|_\infty = \|T(fe^*)\|_\infty \leq \|e^*\| - 1/n$ , and as  $T^{-1}$  is an isometry, it would then follow that  $\|fe^*\|_\infty \leq \|e^*\| - 1/n$ , which is of course absurd. Since  $\{x \in X : \|U(x)e^*\| = \|e^*\|\} = \bigcap_{n=1}^\infty (X - A_n)$ , our claim in the first paragraph of the proof is established.

Now let  $\{e_n^*\}$  be a countable dense subset of  $E_1^*$ . It follows from what we have established in this proof and the previous one that, for each  $n$ , we have that  $\|U(x)e_n^*\| = \|e_n^*\|$  and  $x \rightarrow U(x)e_n^* = (T(e_n^*))(x)$  is norm continuous on a  $G_\delta$  which is dense in  $X$ . Hence if we define  $B$  to be the intersection of such

$G_\delta$ 's for all  $n$  then  $B$  is a dense  $G_\delta$  in  $X$  such that for all  $x \in B$

$$\|U(x)e_n^*\| = \|e_n^*\|$$

and

$$x \rightarrow U(x)e_n^* \text{ is continuous in the norm topology}$$

for all  $n$  [7, p. 200]. The boundedness, for fixed  $x$ , of  $U(x)$  as an operator and the density of  $\{e_n^*\}$  then show that  $U(x)$  is an isometry for  $x \in B$ . And the fact that  $x \rightarrow U(x)e^*$  is the uniform limit of  $x \rightarrow U(x)e_{n_k}^*$  when  $e_{n_k}^* \rightarrow e^*$  shows that  $U(\cdot)$  is continuous in the strong operator topology at points of  $B$ .

Thus all has been established except for the surjectivity of  $U(x)$ . To this end let  $\{v_n^*\}$  be a countable dense subset of  $E_2^*$  and let  $F_n = T^{-1}(v_n^*)$ . We know that each  $F_n$  is norm-continuous on a set  $G_{F_n}$  which is a dense  $G_\delta$ . If we let  $G$  be the dense  $G_\delta$  given by  $G := B \cap (\bigcap_n G_{F_n})$  then at each point  $x \in G$  and for each  $n$  we have

$$v_n^* = (TT^{-1}v_n^*)(x) = (TF_n)(x) = T(F_n(x)) = U(x)F_n(x)$$

so that  $U(x)E_1^*$  contains a dense subset of  $E_2^*$  for all  $x \in G$ . Hence for such  $x$  we have  $U(x)E_1^* = E_2^*$  and the proof of the theorem is complete.

It follows that for Banach duals  $E_1^*, E_2^*$  satisfying the hypotheses of Theorems 2 and 3, and for arbitrary compact Hausdorff spaces  $X_1, X_2$ , if  $T$  is any surjective isometry  $T : C(X_1, (E_1^*, \sigma^*)) \rightarrow C(X_2, (E_2^*, \sigma^*))$  and  $F \in C(X_1, (E_1^*, \sigma^*))$  then there is a dense  $G_\delta$  in  $X_2$  upon which  $TF$  can be described every bit as completely as in the most favorable case for isometries of norm-continuous functions. For arbitrary  $X_1, X_2$  this is the most we can say even if  $E_1^* = E_2^*$  reduces to separable Hilbert space, as shown in the example of [3, p. 36 (C)]: i.e., this description cannot be extended to all of  $X_2$ . However, when the  $X_i$  are metric the situation is considerably nicer.

**Theorem 4.** *Let  $X_i$  be compact metric spaces, let  $E_i^*$  be Banach duals with  $Z(E_i^*) = \mathbf{K}$  for  $i = 1, 2$ , and let  $T : C(X_1, (E_1^*, \sigma^*)) \rightarrow C(X_2, (E_2^*, \sigma^*))$  be a surjective isometry. Then there exists a homeomorphism  $\Phi$  of  $X_2$  onto  $X_1$  and a function  $U(\cdot) : X_2 \rightarrow \mathcal{B}(E_1^*, E_2^*)$  such that for every  $F \in C(X_1, (E_1^*, \sigma^*))$  we have*

$$(1') \quad (TF)(x) = U(x)(F \circ \Phi)(x)$$

for all  $x \in X_2$  where, for all  $x$ ,  $U(x)$  is a surjective isometry.

If, in addition,  $E_2^*$  (and hence  $E_1^*$ ) has property (\*\*), then  $U(\cdot)$  is continuous when  $\mathcal{B}(E_1^*, E_2^*)$  is given its strong operator topology.

*Proof.* Again, because of Theorem 1, we may assume that  $X_1 = X_2 = X$ , and that  $T$  is a  $C(X)$ -module isomorphism. We note, first of all, that if  $x_0 \in X$  and if  $F, G$  are two elements of  $C(X, (E_1^*, \sigma^*))$  such that  $F(x) = G(x)$  for all  $x$  belonging to some neighborhood  $V$  of  $x_0$ , then  $(TF)(x_0) = (TG)(x_0)$ . For let  $f$  be an element of  $C(X)$  such that  $f : X \rightarrow [0, 1]$ , the support of  $f$

is contained in  $V$ , and  $f(x_0) = 1$ . Then  $0 = f(F - G)$  so that  $f(TF) = T(fF) = T(fG) = fT(G)$ . Evaluating this equality at  $x_0$  then gives the desired result.

We next claim that if  $x_0 \in X$  then for every  $F \in C(X, (E_1^*, \sigma^*))$  we have  $\|(TF)(x_0)\| = \|F(x_0)\|$ . First, if  $x_0$  is an isolated point we have  $\|F(x_0)\| = \|\chi_{\{x_0\}}F\|_\infty = \|T(\chi_{\{x_0\}}F)\|_\infty = \|\chi_{\{x_0\}}(TF)\|_\infty = \|(TF)(x_0)\|$ . Hence suppose that  $x_0$  is not an isolated point and take a sequence of distinct points  $\{x_n\}$  such that  $x_n \neq x_0$  for any  $n$  and  $x_n \rightarrow x_0$ . For each  $n = 1, 2, \dots$  let  $V_n$  be a neighborhood of  $x_n$  such that  $\overline{V}_n \cap (\bigcup_{k \neq n} V_k)^- = \emptyset$ . First suppose  $F(x_0) = 0$ . For each  $n$  let  $W_n$  be an open neighborhood of  $x_n$  with  $\overline{W}_n \subseteq \text{Int}(V_n)$  and let  $\varphi_n$  be a continuous function mapping  $X$  to  $[0, 1]$  which is 1 on  $W_n$  and 0 on the complement of  $V_n$ . Then  $F_1 : X \rightarrow E_1^*$  defined as the pointwise sum  $F_1 := \sum_n \varphi_{2n}F$  is an element of  $C(X, (E_1^*, \sigma^*))$ . Thus, if limits are taken in the weak\* topology, we have  $(TF_1)(x_0) = \lim_n (TF_1)(x_{2n+1})$ , which, since  $F_1 = \mathbf{0}$  on  $W_{2n+1}$  is, by the first paragraph of this proof, equal to  $(T(\mathbf{0}))(x_0) = 0$ . But we also have  $(TF_1)(x_0) = \lim_n (TF_1)(x_{2n})$  which, since  $F_1 = F$  on  $W_{2n}$ , is equal to  $(TF)(x_0)$ . Thus  $(TF)(x_0) = 0$ . Consequently, since for any  $F \in C(X, (E_1^*, \sigma^*))$  the function  $F - \mathbf{F}(x_0)$  is zero at  $x_0$ , it follows that

$$(2) \quad (TF)(x_0) = T(\mathbf{F}(x_0))(x_0).$$

Hence one has

$$(3) \quad \|(TF)(x_0)\| \leq \|F(x_0)\|.$$

And if we now replace  $F$  by  $TF$  and  $T$  by  $T^{-1}$  we have

$$(4) \quad \|F(x_0)\| \leq \|(TF)(x_0)\|$$

so that (3) and (4) together give  $\|(TF)(x_0)\| = \|F(x_0)\|$ . Thus if  $U(\cdot)$  is defined as in the proof of Theorem 2 we have

$$(5) \quad (TF)(x_0) = T(\mathbf{F}(x_0)) = U(x_0)F(x_0)$$

and  $\|U(x_0)F(x_0)\| = \|(TF)(x_0)\| = \|F(x_0)\|$  so that  $U(x_0)$  is isometric at all points  $x_0 \in X$ . It is also surjective since the vectors  $(TF)(x_0)$ ,  $F \in C(X, (E_1^*, \sigma^*))$ , exhaust  $E_2^*$ , and (5), together with our assumptions of the first paragraph, gives (1').

To establish the last assertion of the theorem suppose that  $E_2^*$  has property (\*\*). We have to show that if  $e^* \in E_2^*$  and  $\{x_n\} \subseteq X$ ,  $x_n \rightarrow x_0$  then  $U(x_n)e^* \rightarrow U(x_0)e^*$  in the norm topology of  $E_2^*$ . We may suppose that  $\|e^*\| = 1$ . Then, since all the  $U(x)$  are isometric,  $\|U(x_0)e^*\| = 1 = \|U(x_n)e^*\|$  for all  $n$ . We have  $U(x_n)e^* = (T(e^*))(x_n)$  and  $U(x_0)e^* = (T(e^*))(x_0)$  with  $\lim (T(e^*))(x_n) = (T(e^*))(x_0)$  in the weak\* topology of  $E_2^*$ . Thus by our assumption  $(T(e^*))(x_n) \rightarrow (T(e^*))(x_0)$  in norm, and we are done.

*Remarks.* 1. Examination of the proof of Theorem 4 shows that the results depend solely upon the fact that each point of  $X$  is a  $G_\delta$ . Hence the conclusions

of the theorem also hold for those first countable spaces  $X_i$  which fail to be metrizable.

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#### REFERENCES

1. E. Behrends, *M-structure and the Banach-Stone theorem*, Lecture Notes in Mathematics 736, Springer-Verlag, Berlin-Heidelberg-New York, 1979.
2. —, *On the geometry of spaces of  $C_0K$ -valued operators*, Studia Math. 90 (1988), 135–151.
3. M. Cambern and P. Greim, *Mappings of continuous functions on hyperstonean spaces*, Acta Univ. Carolinae, Math. Phys. 28 (1987), 31–40.
4. N. Dunford and J. T. Schwartz, *Linear operators, Part I*, Interscience, New York, 1958.
5. R. Engelking, *General topology*, Monografie Matematyczne, Polish Scientific Publishers, Warszawa, 1977.
6. M. Jerison, *The space of bounded maps into a Banach space*, Ann. of Math. (2) 52 (1950), 309–327.
7. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.
8. I. Namioka, *Separate continuity and joint continuity*, Pacific J. Math. 51 (1974), 515–531.
9. I. Namioka and R. R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. 42 (1975), 735–750.

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