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ON THE POPOV-POMMERENING CONJECTURE FOR GROUPS OF TYPE A_n

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ABSTRACT. The present paper gives an affirmative answer to the Popov-Pommerening conjecture in the case where the reductive group G in the conjecture is of type A_n with $n \leq 4$, and provides a subgroup H of $GL_5(k)$ such that the algebra A^H is finitely generated, but is not spanned by the invariant standard bitableaux.

1. INTRODUCTION

Let k be an algebraically closed field, G an (affine) algebraic group over k, and X an affine k-variety on which G acts rationally (such an X is called a G-variety). Let $k[X]$ be the affine algebra for X. So we have an induced action of G on $k[X]$: $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for $g \in G$, $f \in k[X]$ and $x \in X$. The invariant subalgebra is denoted by $k[X]^G = \{f \in k[X] | g \cdot f = f$ for all $g \in G\}$. An important question in invariant theory is:

$(*)$ Is $k[X]^G$ always a finitely generated k-algebra?

For reductive groups G, the answer to $(*)$ is affirmative due to Mumford, Nagata and Haboush ([M], [N_1] and [H]). (The case $\operatorname{char} k = 0$ goes back to Hilbert and Weyl.) When G is not reductive, then $(*)$ is false thanks to a theorem of Popov [Pp], whose proof is based on counterexamples of Nagata [N_2].

However, interest in the finite generation of invariant subalgebras under various special nonreductive group actions still remains. Weitzenböck proved that $(*)$ is true if G is the additive group $G_a(k)$ of k, $\operatorname{char} k = 0$ and X is an affine k-space (cf. [W]). Seshadri's modern proof (see [Se]) of Weitzenböck's theorem inspired many later results in this direction. Hochschild, Mostow and Grosshans proved that if G is reductive, H is the unipotent radical of some parabolic subgroup of G and G acts rationally on X, then $k[X]^H$ is finitely generated (see [HM], [G_1] and [G_2]).

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The above results led Popov and Pommerening to formulate (independently) the following:

Conjecture. *Let G be a reductive group over k, and let H be a subgroup normalized by a maximal torus. Then for any affine G-variety X, $k[X]^H$ is finitely generated.*

Subgroups H with $k[X]^H$ finitely generated for all affine G-varieties X are called *Grosshans subgroups* (cf. [G_1], [G_2], [G_3], [Po_1], [Po_2], [Po_3]).

Remarks.

(1) Note that the aforementioned results are all special cases of this conjecture.

(2) In the above conjecture, we may assume that G is semi-simple, simply connected and simple, and H is closed and connected (cf. [T_1] and [T_2]).

(3) In the above conjecture, we may assume that $H = \prod_{\alpha\in S} U_\alpha$ where U_α's are root subgroups corresponding to root α, S is a quasi-closed subset (which is closed when the root system is A_n, cf. [T_2]) of the set of positive roots Φ^+ relative to some Borel subgroup and the product is taken in any fixed order since every closed, connected subgroup of G normalized by a maximal torus is generated by certain U_α's ($\alpha \in S$) for some S, and S may be assumed a subset of Φ^+, for the reductive subgroups have the finite generation property. (cf. [T_1] and [T_2])

Recently, the validity of more cases of the conjecture was discovered. The author proved that if $w \in W$, the Weyl group of G, is a product of distinct simple reflections and $S = \{\alpha > 0 | w\alpha > 0\}$, then $H = \prod_{\alpha\in S} U_\alpha$ is a Grosshans subgroup (see [T_1], the proof there is lengthy though). Grosshans proved a more general result independently in [G_3] for $H = \prod_{\alpha\in S} U_\alpha \subset G$, where S is a subset of Φ^+ and $\Phi^+ \backslash S$ is a linearly independent set over $\mathbf{Q}$.

On the other hand, Pommerening studied some classes of unipotent subgroups of GL_n that are Grosshans, by showing that their algebras of invariants are spanned by invariant standard bitableaux (see [Po_1], [Po_2]).

In the present paper, we will answer some open questions (Corollary and Proposition in §2) asked in [Po_2] and [Po_3].

2. The main results

Theorem. *The Popov–Pommerening Conjecture is true if G is a reductive group of type A_n with $n \leq 4$.*

Corollary. *All subgroups of $GL_n(k)$, $SL_n(k)$, $PSL_n(k)$ ($n \leq 5$) normalized by a maximal torus (in the corresponding group) are Grosshans.*

As a corollary of our proof below, we get the following interesting

Proposition. *There exists a Grosshans subgroup H of $GL_5(k)$ such that the invariant subalgebra A^H (where A is a "letter place algebra") is not spanned by the invariant standard bitableaux.*

3. Proof of the theorem

In this section, we prove that the subgroup

$$H = \left\{ \begin{bmatrix} 1 & 0 & a & b & 0 \\ & 1 & 0 & c & d \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix} \middle| a, b, c, d \in k \right\}$$

and (hence) all its conjugates are Grosshans subgroups of $GL_5(k)$. Other subgroups of $GL_5(k)$ $(n \leq 5)$ normalized by a maximal torus can be checked in the same spirit. (In fact, they can be proved to be Grosshans using various known results, cf. $[G_3]$, $[Po_2]$, [T], etc. $[Po_2]$ gives a more detailed account on this.)

First, we need the following "codimension 2" criterion of Grosshans.

Grosshans Criterion. ($[G_1]$, $[G_2]$). Let G be a reductive group and H an observable subgroup. (Recall that a subgroup H of G is called observable if $H = \{g \in G | f(g \cdot x) = f(x))$ for all $x \in G$ and $f \in k[X]^H\}$. See $[G_1]$ for other equivalent definitions and more properties of observable subgroups.) Then the following conditions are equivalent:

(i) H is a Grosshans subgroup of G, i.e., for any affine G-variety X, $k[X]^H$ is finitely generated;

(ii) $k[G]^H$ is a finitely generated k-algebra, where H acts via right translations;

(iii) there is a finite-dimensional rational G-module V and some $v \in V$, so that $H = \mathrm{stab}(v)$ (which denotes the set $\{g \in G | g \cdot v = v\}$), the homogeneous space $G/H \cong G \cdot v$ and $\dim(\overline{G \cdot v} \backslash G \cdot v) \leq \dim(G \cdot v) - 2$ (where $\overline{G \cdot v}$ is the closure of $G \cdot v$ in Zariski topology), adopting the convention that $\dim(\varnothing) = -2$.

We will denote $\overline{G \cdot v} \backslash G \cdot v$ by $\partial(G \cdot v)$ and $\dim(G \cdot v) - \dim(\partial(G \cdot v))$ by $\mathrm{codim}_{\overline{G \cdot v}} \partial(G \cdot v)$.

Let $\alpha, \beta, \gamma, \delta$ be the simple roots in A_4 with the Dynkin diagram

$$\alpha \quad \beta \quad \gamma \quad \delta$$

and $\lambda_\alpha, \lambda_\beta, \lambda_\gamma, \lambda_\delta$ be the corresponding fundamental dominant weights. Denote by $V(\lambda)$ an irreducible representation for G of highest weight λ, and by $v_\mu = v_\mu(\lambda)$ a weight vector of weight μ in $V(\lambda)$. By Grosshans criterion, it suffices to show that for

$$V = V(\lambda_\beta) \oplus V(\lambda_\delta) \oplus V(\lambda_\delta) \oplus V(\lambda_\beta)$$

and

$$v = v_{\lambda_\beta - \beta} \oplus v_{\lambda_\delta} \oplus v_{\lambda_\delta - \delta - \gamma} \oplus v_{\lambda_\beta - \beta - \alpha - \gamma - \delta}$$

in V, the subgroup H defined at the beginning of this section is $\mathrm{Stab}(v)$ and $\mathrm{codim}_{\overline{G\cdot v}}\,\partial(G\cdot v)\geq 2$.

Note that all the weights appearing in the summands of v are of multiplicity 1, being "extreme weights" (in the W-orbits of the highest weights). Thus, the expression is unambiguous, since replacing a summand of v by a nonzero multiple of it gives rise to isomorphic $G\cdot v$ and $\overline{G\cdot v}$ respectively.

That $H=\mathrm{Stab}(v)$ follows directly from the following lemma, whose proof is easy (cf. [T_1]) and is skipped here.

Lemma 1. *Let $V(\lambda)$ be a rational G-module of highest weight λ with a highest weight vector v_λ, U_α the root subgroup corresponding to $\alpha\in\Phi$, and $w\in W$. Then U_α fixes $w\cdot v_\lambda$ if and only if $(\alpha, w\lambda)\geq 0$.*

To verify the "codimension 2" condition, we use the following

Lemma 2. *Let G be an arbitrary algebraic group, V and affine G-variety (in particular a G-module), $v\in V$ and B any parabolic (in particular Borel) subgroup. Then*

$$\partial(G\cdot v)=G\cdot(\overline{B\cdot v}\backslash G\cdot v)\subseteq G\cdot(\partial(B\cdot v)).$$

The second inclusion is trivial, while the first equality is a consequence of the completeness of G/B (cf. §2.13 of [St]).

Next we investigate $B\cdot v$ and $\partial(B\cdot v)$.

Write $B=UT$ and $U=U_{\alpha+\beta+\gamma+\delta}U_{\gamma+\delta}U_\delta U_\gamma U_\beta U_\alpha H$. By the way in which the root subgroups act (cf. §3.3 of [St]), we have that a typical element in $B\cdot v$ is of the form

$$\begin{aligned}&[av_{\lambda_\beta-\beta}+a_1v_{\lambda_\beta}]\oplus bv_{\lambda_\delta}\oplus[cv_{\lambda_\delta-\delta-\gamma}+c_1v_{\lambda_\delta-\delta}+c_2v_{\lambda_\delta}]\\&\oplus[dv_{\lambda_\beta-\beta-\alpha-\gamma-\delta}+d_1v_{\lambda_\beta-\beta-\gamma-\delta}+d_2v_{\lambda_\beta-\beta-\gamma}\\&+d_3v_{\lambda_\beta-\beta-\alpha-\gamma}+d_4v_{\lambda_\beta-\beta-\alpha}+d_5v_{\lambda_\beta-\beta}+d_6v_{\lambda_\beta}],\end{aligned}$$

where $a, b, c, d\in k^*$, $a_i, c_i, d_i\in k$, and the coefficients satisfy the following equations;

$$\begin{array}{ll}\text{(i)} & dd_2=d_1d_3,\\ \text{(ii)} & d_1d_4=dd_5,\\ \text{(iii)} & d_1c_2=c_1d_2+cd_5,\\ \text{(iv)} & dc_2=c_1d_3+cd_4.\end{array}$$

Thus, the points in $\overline{B\cdot v}$ have the same form but a, b, c, d may be equal to zero (i.e., $\in k$ instead of k^*), and (i)–(iv) still hold. Consequently

$$\partial(B\cdot v)=(\overline{B\cdot v}\cap\mathscr{V}(a))\cup(\overline{B\cdot v}\cap\mathscr{V}(b))\cup(\overline{B\cdot v}\cap\mathscr{V}(c))\cup(\overline{B\cdot v}\cap\mathscr{V}(d)),$$

where $\mathscr{V}(f)$ denotes the set of points at which f is 0. $O(f)$ will denote the complement of $\mathscr{V}(f)$.

By abuse of notation, for a $v'=[av_{\lambda_\beta-\beta}+a_1v_{\lambda_\beta}]\oplus\cdots\in\overline{B\cdot v}$, we will use a, a_1, etc. both as the coordinate function and the coefficients of the vector $v_{\lambda_\beta-\beta}, v_{\lambda_\beta}$ etc. respectively. So we have $a(v')=a, a_1(v')=a_1, \ldots$.

Let $v' \in \partial(B \cdot v)$ and $a(v') = 0$, $b(v') \neq 0$, $c(v') \neq 0$, $d(v') \neq 0$. Then by the way U_α's and T act, we can choose a proper $g \in G$ (in fact in B), such that $b(g \cdot v') = c(g \cdot v') = d(g \cdot v') = 1$ and all $c_i(g \cdot v')$, $d_i(g \cdot v')$ are 0. Thus,

$$g \cdot v' = a_1 v_{\lambda_\beta} \oplus v_{\lambda_\delta} \oplus v_{\lambda_\delta - \delta - \gamma} \oplus v_{\lambda_\beta - \beta - \alpha - \gamma - \delta}.$$

By direct computation, $\mathrm{Stab}(g \cdot v')$ contains

$$U_{-\alpha}, U_\beta, U_{-\gamma}, U_{\alpha+\beta} U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{\beta+\gamma+\delta},$$

so $\dim(G \cdot v') \leq \dim(G \cdot v) - 3$.

The above argument shows that

$$\begin{aligned} &G \cdot (\partial(B \cdot v) \cap \mathscr{V}(a) \cap O(b) \cap O(c) \cap O(d)) \\ &\quad \subseteq \bigcup_{a_1 \in k} G \cdot (a_1 v_{\lambda_\beta} \oplus v_{\lambda_\delta} \oplus v_{\lambda_\delta - \delta - \gamma} \oplus v_{\lambda_\beta - \beta - \alpha - \gamma - \delta}). \end{aligned}$$

The latter, hence the former, has dimension $\leq \dim(G \cdot v) - 2$.

By the same kind of arguments, we can show that $\dim G \cdot [\partial(B \cdot v) \cap O(d)] \leq \dim(G \cdot v) - 2$. (Check out that $G \cdot [\partial(B \cdot v) \cap O(a) \cap O(b) \cap \mathscr{V}(c) \cap O(d)]$, $G \cdot [\partial(B \cdot v) \cap \mathscr{V}(a) \cap O(b) \cap \mathscr{V}(c) \cap O(d)]$ etc. have the required dimension.) Therefore, it suffices to show that $\dim G \cdot (\partial(B \cdot v) \cap \mathscr{V}(d)) \leq \dim G \cdot v - 2$.

Let $v' \in \partial(B \cdot v) \cap \mathscr{V}(d) \cap O(d_1)$. By (ii) above, $d_3 = 0$; by (iii) above, $d_4 = 0$. (In other words, $\partial(B \cdot v) \cap \mathscr{V}(d) \cap O(d_1) \cap O(d_3) = \varnothing$, etc.) By replacing v' by $g \cdot v'$ for some $g \in B$ as before, we may assume that $d_2(v') = d_5(v') = d_6(v') = 0$.

Case 1. $c(v') \neq 0$. Then $c_2 = 0$ by (iv) and $c_1 = 0$ (apply some element in U_γ if necessary). Therefore, if $a \neq 0$, then we may assume

$$v' = a v_{\lambda_\beta - \beta} \oplus v_{\lambda_\delta} \oplus v_{\lambda_\delta - \delta - \gamma} \oplus v_{\lambda_\beta - \beta - \gamma - \delta},$$

and $\mathrm{Stab}(v')$ contains $U_\alpha, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{-(\beta+\gamma)}, T \cap G_{\beta+\gamma}$ (where $G_{\beta+\gamma}$ is the group generated by $U_{\beta+\gamma}$ and $U_{-(\beta+\gamma)}$); if $a = 0$, then we may assume

$$v' = v_{\lambda_\beta} \oplus v_{\lambda_\delta} \oplus v_{\lambda_\delta - \delta - \gamma} \oplus v_{\lambda_\beta - \beta - \gamma - \delta},$$

and $\mathrm{Stab}(v')$ contains $U_\alpha, U_\beta, U_{-\gamma}, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{\alpha+\beta+\gamma+\delta}$, in better shape. Thus

$$\dim G \cdot (\partial(B \cdot v) \cap \mathscr{V}(d) \cap O(d_1) \cap O(c)) \leq \dim G \cdot v - 2.$$

Case 2. $c(v') = 0$. Here, according to the two subcases $c_1 \neq 0$ and $c_1 = 0$, we can carry out similar computation. The detail is skipped.

The above shows that $\dim G \cdot (\partial(B \cdot v) \cap \mathscr{V}(d) \cap O(d_1)) \leq \dim G \cdot v - 2$. For $G \cdot (\partial(B \cdot v) \cap \mathscr{V}(d) \cap \mathscr{V}(d_1))$, the "codimension 2" condition is checked out analogously using (i)–(iv) above, and the detail again is omitted.

4. Some remarks

(a) The V and v constructed in the proof in §3 is not canonical, of course. For example, if we take $V = V_{\lambda_\beta} \oplus V_{\lambda_\delta} \oplus V_{\lambda_\delta} \oplus V_{\lambda_\beta} \oplus V_{\lambda_\gamma}$, and $v = v_{\lambda_\beta - \beta} \oplus v_{\lambda_\delta} \oplus v_{\lambda_\delta - \delta - \gamma} \oplus v_{\lambda_\beta - \beta - \alpha - \gamma - \delta} \oplus v_{\lambda_\gamma - \gamma}$, the proof still goes through.

(b) The proposition in §2 is verified by taking H to be the subgroup given in §3, since it is known that this A^H is not spanned by the invariant standard bitableaux (see [Po_2]).

(c) The proposition therefore shows that Pommerening's approach (through invariant standard bitableaux) to the conjecture for $GL_n(k)$ is limited. While ours is very involved and somewhat ad hoc, a more conceptual approach is welcome.

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