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COFINAL FAMILIES OF COMPACT SUBSETS OF AN ANALYTIC SET

HOWARD BECKER

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ABSTRACT. We consider a question of van Douwen regarding the minimum cardinality of cofinal families of compact sets in certain topological spaces. We show that the question cannot be answered in ZFC.

The cardinal $\mathfrak{d}$ is defined as follows.

$$\mathfrak{d} = \min\{|\mathscr{D}| : \mathscr{D} \subset \omega^\omega \text{ and for all } f \in \omega^\omega, \text{ there is a } g \in \mathscr{D} \text{ such that for all but finitely many } n,\ g(n) > f(n)\}.$$

Clearly $\aleph_1 \leq \mathfrak{d} \leq \mathfrak{c}$, and $\mathrm{cof}(\mathfrak{d}) > \omega$. Hechler [5] showed that nothing else about the size of $\mathfrak{d}$ can be proved in ZFC.

For any topological space X, let $\mathscr{K}(X)$ be the space of all non-empty compact subsets of X, with the Vietoris topology. Define the cardinal $\mathrm{cof}(\mathscr{K}(X))$ as follows.

$$\mathrm{cof}(\mathscr{K}(X)) = \min\{|\mathscr{L}| : \mathscr{L} \subset \mathscr{K}(X) \text{ and for all compact } K \subset X, \text{ there is an } L \in \mathscr{L} \text{ such that } K \subset L\}.$$

Note that $\mathfrak{d} = \mathrm{cof}(\mathscr{K}(\omega^\omega))$, where ω^ω has the usual topology.

Van Douwen [2] considered the problem of calculating $\mathrm{cof}(\mathscr{K}(X))$ for various separable metrizable spaces X. He showed that for any such X, if X is analytic ($\mathbf{\Sigma}^1_1$) but not σ-compact, then $\mathrm{cof}(\mathscr{K}(X)) \geq \mathfrak{d}$. He asked ([1], [2, 8.11]) two questions: If X is separable and metrizable and analytic (or at least absolutely Borel), is $\mathrm{cof}(\mathscr{K}(X)) \leq \mathfrak{d}$? I thank Peter Nyikos for bringing these questions to my attention.

Van Engelen [3] gave a positive answer to the *Borel* question. In fact, he proved a stronger statement: If X is any coanalytic ($\mathbf{\Pi}^1_1$) set in a Polish space, then $\mathrm{cof}(\mathscr{K}(X)) \leq \mathfrak{d}$.

Clearly it is consistent with ZFC that van Douwen's *analytic* question also has a positive answer—for example, if the continuum hypothesis holds. The purpose of this paper is to show that a negative answer is also consistent, hence the question cannot be decided in ZFC.

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Theorem. Con(ZFC) *implies* Con$(ZFC+\mathfrak{d}=\aleph_1+$ *There is a* $\mathbf{\Sigma}^1_1$ *set* $X\subset 2^\omega$ *such that* $\mathrm{cof}(\mathscr{K}(X))=\aleph_2=\mathfrak{c})$.

For more information on the topological matters, see van Douwen [2]; on $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ sets, see Moschovakis [9]; on consistency proofs, see Jech [6].

Lemma 1 (Martin-Solovay [8]; see [4, 23J and 23N(g)]). Con(ZFC) *implies* Con$(ZFC+\mathfrak{c}=\aleph_2+$ *Every set of cardinality* $\aleph_1$ *in every Polish space is* $\mathbf{\Pi}^1_1)$.

Lemma 2 (Hechler [5]; see [6, p. 260]). *There exists a notion of forcing which preserves cardinals, preserves the value of* $\mathfrak{c}$, *and makes* $\mathfrak{d}=\aleph_1$.

$$\text{Let } \boldsymbol{\pi}^1_1=\sup\{\alpha\in\mathrm{Ord}\colon \alpha \text{ is the rank of a } \mathbf{\Pi}^1_1 \text{ well-founded relation on } 2^\omega\}.$$

By the Kunen-Martin Theorem (see [9, 2G.4]), $\boldsymbol{\pi}^1_1\le\aleph_2$.

Lemma 3. Con(ZFC) *implies* Con$(ZFC+\mathfrak{d}=\aleph_1+\boldsymbol{\pi}^1_1=\aleph_2=\mathfrak{c})$.

Proof. If every set of cardinality $\aleph_1$ is $\mathbf{\Pi}^1_1$, then there are $\mathbf{\Pi}^1_1$ well-orderings of every order-type less than ω_2, so clearly $\boldsymbol{\pi}^1_1=\aleph_2$. Therefore, by Lemma 1, Con(ZFC) implies Con(ZFC$+\boldsymbol{\pi}^1_1=\aleph_2=\mathfrak{c}$). We next show: Con(ZFC$+\boldsymbol{\pi}^1_1=\aleph_2=\mathfrak{c}$) implies Con(ZFC$+\mathfrak{d}=\aleph_1+\boldsymbol{\pi}^1_1=\aleph_2=\mathfrak{c}$). Let $M\models(\mathrm{ZFC}+\boldsymbol{\pi}^1_1=\aleph_2=\mathfrak{c})$, let P be a notion of forcing satisfying Lemma 2, and let N be P-generic over M. Thus $N\models(\mathrm{ZFC}+\mathfrak{d}=\aleph_1+c=\aleph_2)$. Let $\lambda<\aleph_2^M=\aleph_2^N$; to complete the proof it will suffice to show that $\lambda<(\boldsymbol{\pi}^1_1)^N$. Since $M\models(\boldsymbol{\pi}^1_1=\aleph_2)$, there is, in M, a wellfounded $\mathbf{\Pi}^1_1$ binary relation R on 2^ω of rank at least λ. Let ψ be a Π^1_1-formula with two free variables (and a real parameter from M) which defines R. The statement "The relation ψ is wellfounded" is Π^1_2, so by the Shoenfield Absoluteness Theorem, it is true in N. ψ itself is, of course, also absolute, so $R\subset R'$, where

$$R'=\{(x,y)\in N\times N\colon N\models\psi(x,y)\}.$$

So $\mathrm{rank}(R')\ge\mathrm{rank}(R)$. Hence, in N, there is a $\mathbf{\Pi}^1_1$ wellfounded relation of rank at least λ. That is, $\lambda<(\boldsymbol{\pi}^1_1)^N$. □

For any topological space Y, let $\mathrm{kc}(Y)$ denote the *compact covering number* of Y, that is,

$$\min\{|\mathscr{L}|\colon \mathscr{L}\subset\mathscr{K}(Y) \text{ and } \bigcup\mathscr{L}=Y\}.$$

Lemma 4. *If* $\boldsymbol{\pi}^1_1=\aleph_2$, *then there is a* $\mathbf{\Pi}^1_2$ *set* $Y\subset 2^\omega$ *such that* $\mathrm{kc}(Y)\ge\aleph_2$.

Proof. Let $U\subset 2^\omega\times(2^\omega\times 2^\omega)$ be universal for $\mathbf{\Pi}^1_1$ subsets of $2^\omega\times 2^\omega$, i.e., U is $\mathbf{\Pi}^1_1$ and every $\mathbf{\Pi}^1_1$ set in $2^\omega\times 2^\omega$ is equal to $U_x=\{(y,z)\colon(x,y,z)\in U\}$ for some x. Let

$$Y=\{x\in 2^\omega\colon \text{The binary relation } U_x \text{ is wellfounded}\}.$$

Y is $\mathbf{\Pi}^1_2$. Clearly $\sup\{\operatorname{rank}(U_x) : x \in Y\} = \boldsymbol{\pi}^1_1 = \aleph_2$. We claim that for any $\mathbf{\Pi}^1_1$ set $Z \subset Y$, $\sup\{\operatorname{rank}(U_x) : x \in Z\} < \boldsymbol{\pi}^1_1$. Assuming this claim, for any collection $\mathscr{L}$ of $\aleph_1$ compact (or even $\mathbf{\Pi}^1_1$) subsets of Y, $\sup\{\operatorname{rank}(U_x) : x \in \bigcup\mathscr{L}\} < \aleph_2$ hence $\mathscr{L}$ cannot cover Y. To prove the claim, let $Z \subset Y$ be $\mathbf{\Pi}^1_1$. Define $R \subset (2^\omega \times 2^\omega) \times (2^\omega \times 2^\omega)$ as follows:

$$((x,y),(x',y')) \in R \text{ iff } [x = x' \text{ and } x \in Z \text{ and } (x,y,y') \in U].$$

R is a $\mathbf{\Pi}^1_1$ binary relation on $2^\omega \times 2^\omega$, and it is wellfounded. It is easy to see that the rank of R is $\sup\{\operatorname{rank}(U_x) : x \in Z\}$. Since R is $\mathbf{\Pi}^1_1$, this rank must be less than $\boldsymbol{\pi}^1_1$. □

Lemma 5 (van Engelen [3]). *For any $X \subset 2^\omega$, $\operatorname{cof}(\mathscr{K}(X)) = \operatorname{kc}(\mathscr{K}(X))$.*

Let $\mathbf{\Gamma}$ be either $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_2$, and let E be a 0-dimensional compact metrizable space. A pointset $Y \subset E$ is called $\mathbf{\Gamma}$-*hard* if for every 0-dimensional compact metrizable space E', and every $\mathbf{\Gamma}$ set $Z \subset E'$, there is a continuous function $f : E' \to E$ with $Z = f^{-1}[Y]$. A set is $\mathbf{\Gamma}$-*complete* if it is a $\mathbf{\Gamma}$ set which is $\mathbf{\Gamma}$-hard. Note that $\mathbf{\Gamma}$-complete sets exist; for example, any universal set is complete.

Lemma 6 (Kechris-Louveau-Woodin [7, p. 266]). *For any $X \subset 2^\omega$, if X is $\mathbf{\Sigma}^1_1$-hard, then $\mathscr{K}(X)$, considered as a pointset in the space $\mathscr{K}(2^\omega)$, is $\mathbf{\Pi}^1_2$-hard.*

Lemma 7. *For any cardinal κ, if there exists a $\mathbf{\Pi}^1_2$ set $Y \subset 2^\omega$ with $\operatorname{kc}(Y) \geq \kappa$, then there exists a $\mathbf{\Sigma}^1_1$ set $X \subset 2^\omega$ with $\operatorname{cof}(\mathscr{K}(X)) \geq \kappa$.*

Proof. Let $Y \subset 2^\omega$ be $\mathbf{\Pi}^1_2$ with $\operatorname{kc}(Y) \geq \kappa$. Let $X \subset 2^\omega$ be any $\mathbf{\Sigma}^1_1$-complete set. By Lemma 6, $\mathscr{K}(X)$ is $\mathbf{\Pi}^1_2$-hard, so there exists a continuous $f : 2^\omega \to \mathscr{K}(2^\omega)$ such that $f^{-1}[\mathscr{K}(X)] = Y$. If $\mathscr{L} \subset \mathscr{K}(\mathscr{K}(X))$ covers $\mathscr{K}(X)$, then $\{f^{-1}[K] : K \in \mathscr{L}\}$ covers Y, so $\operatorname{kc}(\mathscr{K}(X)) \geq \operatorname{kc}(Y)$. By Lemma 5, $\operatorname{cof}(\mathscr{K}(X)) = \operatorname{kc}(\mathscr{K}(X))$; hence $\operatorname{cof}(\mathscr{K}(X)) \geq \kappa$. □

Lemma 8. *If $\boldsymbol{\pi}^1_1 = \aleph_2$, then there is a $\mathbf{\Sigma}^1_1$ set $X \subset 2^\omega$ such that $\operatorname{cof}(\mathscr{K}(X)) \geq \aleph_2$.*

Proof. Lemmas 4 and 7. □

The theorem follows from Lemmas 3 and 8.

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Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208