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THURSTON NORM MINIMIZING SURFACES AND SKEIN TREES FOR LINKS IN S^3

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ABSTRACT. This paper gives a method for constructing all links in S^3, beginning with the unknot and adding at most one to the norm of the link at each stage. This has two corollaries. The first is that links with 'minimal' skein trees are fibered. The second is a complete list of all links with skein trees of height two.

In [2] it is shown that three links related by the Conway moves have closely related Thurston norm minimizing Seifert surfaces. This is used to give a lower bound on the height of a skein tree for a link. Here these results are extended to yield a method for constructing all links, beginning with the unknot and adding at most one to the norm of the link at each stage. This has two corollaries: the first is that if the lower bound obtained in [2] is realized, then the link must be fibered. The second is a complete list of all links with skein trees of height two. It is surprisingly easy to obtain this second corollary, especially when one considers that the question of characterizing skein trees of height one is equivalent to the question of whether one can band together two knots in some non-trivial way to obtain the unknot. This was a long-standing problem, eventually solved by M. Scharlemann [1].

(1.1) **Definition.** A Seifert surface for an oriented link L in S^3 is an oriented surface S, with no closed components, such that $\partial S = L$. S is *taut* if there is no Seifert surface for L of larger Euler characteristic. This is equivalent to being Thurston norm minimizing and incompressible [2]. Let $X(L)$ be minus the Euler characteristic of a taut Seifert surface for L.

(1.2) **Definition.** Three oriented links L_+, L_- and L_0 in S^3 are *related by the Conway moves* if they are identical outside a three-ball B where they appear as in Figure 1. We refer to [2] for the definitions of a *skein tree* T for a link L, a *node* of T, a *leaf* of T, the *height of a skein tree for* L and the *height of* L, written $h(L)$.

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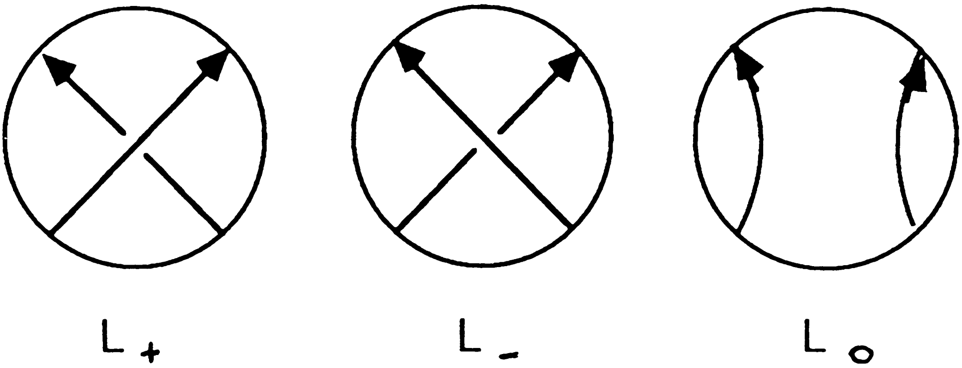

FIGURE 1

(1.3) **Definition.** Let $L = L_{+(-)}$ be a node of a skein tree T, and let $L_{-(+)}$ and L_0 be the links in T obtained from $L_{+(-)}$ by the Conway moves. Call $L_{-(+)}$ the *left branch* of $L_{+(-)}$ and L_0 the *right branch.*

(1.4) **Definition.** Let S be a Seifert surface for the link L, α an arc properly imbedded in S. L' is obtained from L by *twisting along* α if L and L' are related as shown in Figure 2a. L' is obtained from L by *adding a twisted band along* α if L and L' are related as shown in Figure 2b. Note that the operation of adding a twisted band along an arc is the same as plumbing a Hopf band to S along α (see [3]). Also note that both operations yield an obvious Seifert surface S' for L'.

(1.5) **Definition.** A link L in S^3 is *split* if it contains an essential 2-sphere in its complement. Otherwise it is *non-split.*

(1.6) **Theorem.** *Let* L *be a non-split link in* S^3. *Then there exists a sequence of triples* $(L_0, S_0, \alpha_0) \to (L_1, S_1, \alpha_1) \to \cdots \to (L_{m-1}, S_{m-1}, \alpha_{m-1}) \to L_m = L$ *such that:*

1. L_0 *is the unknot.*
2. $m \leq h(L)$.
3. S_i *is a taut Seifert surface for* L_i, $i = 0, \dots, m-1$.
4. α_i *is an arc properly imbedded in* S_i.
5. L_{i+1} *is obtained from* L_i *by either twisting along* α *or by adding a twisted band along* α.

Proof. Let T be a skein tree for L of height $h(L)$. We choose an imbedded path p in T from L to a leaf of T as follows: suppose $L_{+(-)}$ is a node of T, with left branch $L_{-(+)}$ and right branch L_0. If $X(L_{-(+)}) \geq X(L_{+(-)})$, connect $L_{+(-)}$ to $L_{-(+)}$. Otherwise connect $L_{+(-)}$ to L_0. Beginning with L, use this

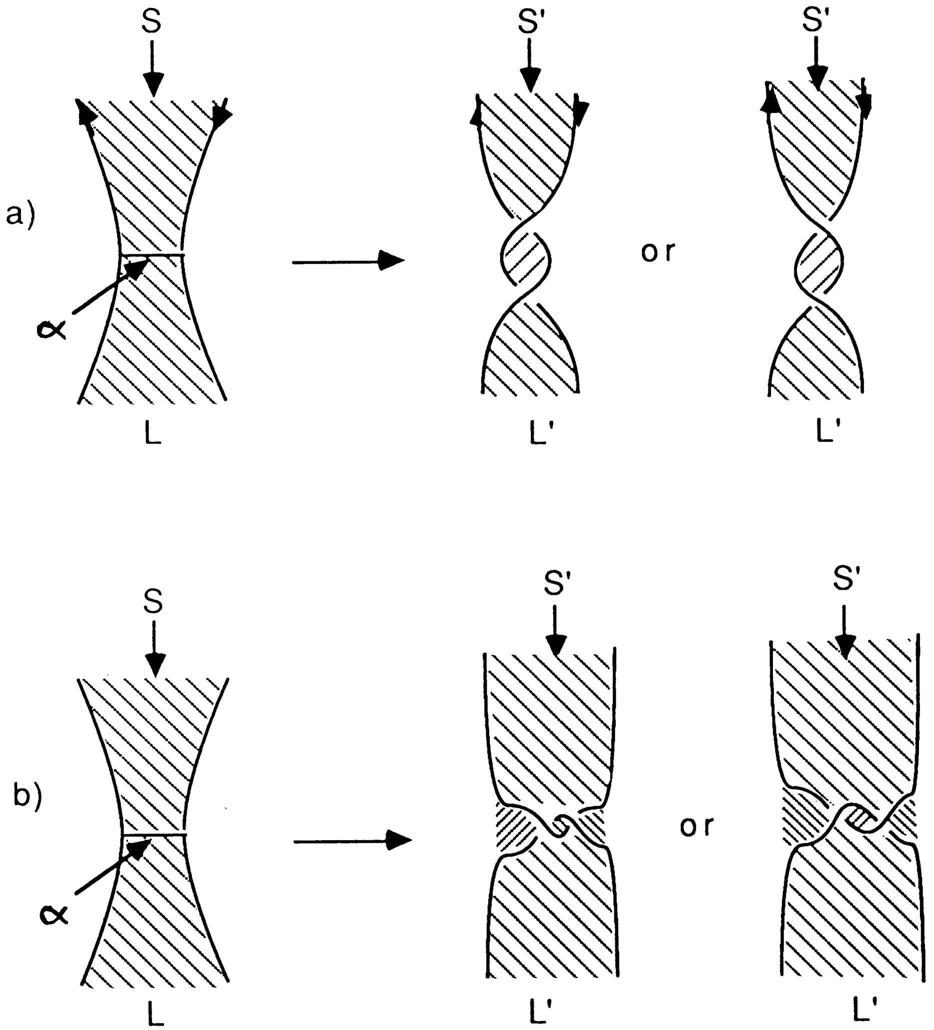

FIGURE 2

rule to construct p. Label the links in p from zero to m, beginning with the leaf of T. Note that $m \leq h(L)$.

Suppose L_i is a right branch of L_{i+1}. Then X (left branch $L_{i+1}) < X(L_{i+1})$. By [2, Proposition 3.1], it follows that there exists a taut Seifert surface S_i for L_i and an arc α_i properly imbedded in S_i such that L_{i+1} can be obtained from L_i by adding a twisted band along α_i.

Suppose L_i is a left branch of L_{i+1}. Then $X(L_i) \geq X(L_{i+1})$. Using the arguments from [2, Theorem 1.4], it follows that there exists a taut Seifert surface S_i for L_i and an arc α_i properly imbedded in S_i such that L_{i+1} can be obtained from L_i by twisting along α_i.

It remains to show that L_0 is indeed the unknot as opposed to an unlink of $j > 1$ components.

Claim. *Let L be a split link in S^3. If S is a taut Seifert surface for L, then one can find a splitting sphere F such that $S \cap F = \varnothing$.*

Proof. Choose F to minimize the number of components in $S \cap F$. If $S \cap F \neq \varnothing$, then an innermost disk argument shows that S is compressible, hence not taut.

From this claim it follows that if L_i in the above construction is split, then so is L_{i+1}, $i = 0, \dots, m-1$. Since we assume that L is not a split link, it follows that L_0 is not split, hence L_0 is the unknot.

(1.7) **Corollary.** *If L is a non-split link in S^3 such that $h(L) = X(L) + 1$, then L is a fibered link.*

Proof. Let L be a link in S^3 with $h(L) = n$ and let

$$(L_0, S_0, \alpha_0) \to (L_1, S_1, \alpha_1) \to \cdots \to (L_{m-1}, S_{m-1}, \alpha_{m-1}) \to L_m = L$$

be a sequence of triples for L constructed as in Theorem 1.6.

Claim. *Then $X(L) \leq m - 1$, and equality holds if and only if L_{i+1} is obtained from L_i by adding a twisted band along α_i for all $i = 0, \dots, m-1$.*

Proof. In the proof of 1.6, L_{i+1} is obtained from L_i by twisting along α_i if and only if $X(L_{i+1}) \leq X(L_i)$. So also we know that if L_{i+1} is obtained from L_i by adding a twisted band along α_i, $X(L_{i+1}) > X(L_i)$. But the Seifert surface S' constructed for L_{i+1} by plumbing a Hopf band to S_i along α_i has $-\chi(S') = -\chi(S_i) + 1 = X(L_i) + 1$, so $X(L_{i+1}) \leq X(L_i) + 1$. Hence $X(L_{i+1}) = X(L_i) + 1$. Since $X(L_0) = -1$, it follows that $X(L) \leq m - 1$, with equality if and only if L_{i+1} is obtained from L_i by adding a twisted band along α_i for all $i = 0, \dots, m-1$.

Hence we know $X(L) + 1 \leq m \leq n$. If $X(L) + 1 = h(L)$, then $m = h(L)$ and L_{i+1} is always obtained from L_i by adding a twisted band along α_i for $i = 0, \dots, m-1 (= h(L) - 1)$. Plumbing a Hopf band to the taut Seifert surface of a fibered knot yields a fibered knot [3], hence L is fibered.

(1.8) **Corollary.** *The following is a complete list of all non-split links of height less than or equal to two (see Figure 3); it is obvious from the proof that there are some restrictions on the orientations of the components in 5). These are indicated in Figure 3.*

1. *the unknot* $(h = 0)$;
2. *the Hopf links* $(h = 1)$;
3. *the right- and left-handed trefoils*;
4. *the figure eight knot*;
5. *the Hopf link with an extra twist*;
6. *a chain of three unknotted components.*

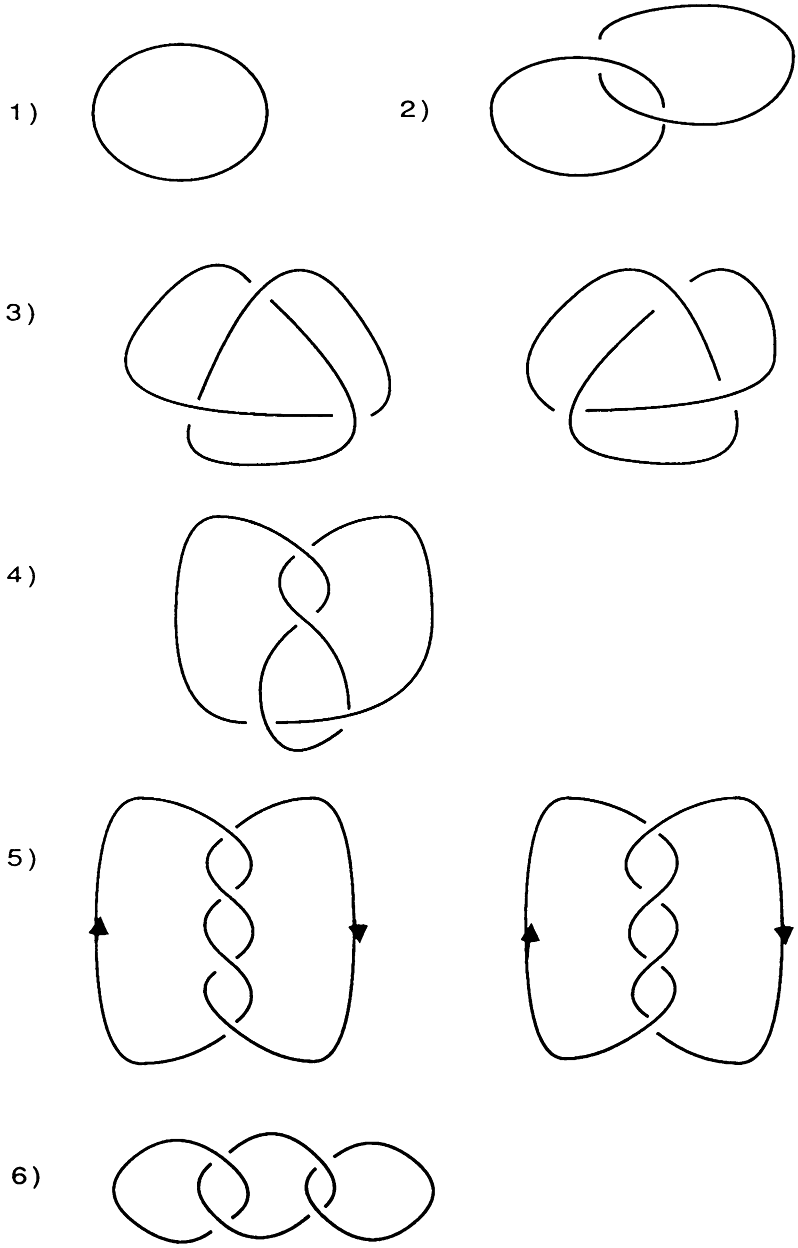

FIGURE 3

Proof. Suppose $h(L) \leq 2$. Apply Theorem 1.6 to obtain a sequence of at most two triples. By the theorem, L_0 is the unknot. If we twist along α_0, we obtain L_1 also equal to the unknot, so without loss of generality we can assume that at the first stage we add a twisted band along α_0. If we add a twisted band along α_0 we obtain L_1 equal to the Hopf link (2). A taut Seifert surface for the Hopf link is an annulus A, unique up to isotopy. The annulus contains only two distinct arcs up to automorphism, one essential and one inessential.

Case (i). Suppose α_1 is an essential arc in A. If we add a twisted band along α_1 we obtain either one of the two trefoils (3) or the figure eight knot (4). If we twist along α_1 we obtain either the trivial link of two components, which is split, or the Hopf link with an extra twist (5).

Case (ii). Suppose α_1 is not essential in A. If we twist along α_1 we again obtain the Hopf link. If we add a twisted band along α_1 we obtain a chain of three unknotted components (6).

This shows that 1–6 are the only non-split links that could possibly have height less than or equal to two. It remains to show that these links indeed all have skein trees of height less than or equal to two. One can easily construct such trees for these links; this is left to the reader.

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