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AUTOMORPHISMS OF EXTENDED CURRENT ALGEBRAS

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ABSTRACT. We construct a (noncentral) extension of current algebras and study the adjoint action induced by the current group.

1. INTRODUCTION

Let us consider the affine Kač-Moody algebra [5] associated to a finite-dimensional linear reductive Lie algebra $\mathbf{g}$,

$$\hat{\mathbf{g}} = \tilde{\mathbf{g}} \oplus \mathbf{Cc} \oplus \mathbf{Cd}, \tag{1}$$

where $\tilde{\mathbf{g}} = \mathbf{C}[t, t^{-1}] \otimes_{\mathbf{C}} \mathbf{g}$. $\hat{\mathbf{g}}$ is a central extension of $\tilde{\mathbf{g}}$ and the bracket is defined as follows $(u, v \in \mathbf{g}; \lambda, \mu, \lambda_1, \mu_1 \in \mathbf{C})$:

$$\begin{aligned}(2)\qquad &[t^k \otimes u \oplus \lambda c \oplus \mu d, t^{k_1} \otimes v \oplus \lambda_1 c \oplus \mu_1 d]^\wedge \\ &= (t^{k+k_1} \otimes [u, v] + \mu k_1 t^{k_1} \otimes v - \mu_1 k t^k \otimes u) \oplus k\delta_{k,-k_1}(u|v)c.\end{aligned}$$

There is a nondegenerate invariant symmetric bilinear form $(\cdot|\cdot)^\wedge$ on $\hat{\mathbf{g}}$ defined by:

$$\begin{aligned}&(t^k \otimes u | t^{k_1} \otimes v)^\wedge = \delta_{k,-k_1}(u|v) \\ &(c|d)^\wedge = (d|c)^\wedge = 1 \\ &\text{others are zero.}\end{aligned} \tag{3}$$

for every $u, v \in \mathbf{g}$ and $k, k_1 \in \mathbf{Z}$. The group $\widetilde{G} = \mathbf{C}[t, t^{-1}] \otimes G$ acts on $\hat{\mathbf{g}}$ by automorphisms [6, 2] as follows:

$$\begin{aligned}&(\widehat{\mathrm{Ad}}g)(u(t)) = gug^{-1}(t) + \mathrm{Res}_0 \operatorname{tr}(ug^{-1}g')c \\ &(\widehat{\mathrm{Ad}}g)c = c \\ &(\widehat{\mathrm{Ad}}g)d = -tg'g^{-1} - \frac{1}{2}\mathrm{Res}_0 \operatorname{tr} t(g'g^{-1})^2 c + d,\end{aligned} \tag{4}$$

for every $g \in \widetilde{G}$ and $u \in \tilde{\mathbf{g}}$; this action preserves the bilinear form $(\cdot|\cdot)^\wedge$. Here $\tilde{\mathbf{g}}$ is the algebra of finite Laurent polynomial maps. By a change of variable

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$t = e^{i\theta}$, each Laurent polynomial map can be realized as a smooth map from S^1 to $\mathbf{g}$. This means that we have an injection from the algebra of Laurent polynomial maps into the loop algebra, the algebra of all smooth maps from S^1 to $\mathbf{g}$. Since our main interest is the latter, we will keep the notation $\tilde{\mathbf{g}}$ for the loop algebra unless otherwise specified. More generally the Lie algebra $\tilde{\mathbf{g}}(M)$ of all smooth maps from an arbitrary manifold M to $\mathbf{g}$ is known as the current algebra [3, 1].

In this paper, we consider a (noncentral) extension of the current algebra and study the automorphisms on the extended algebra induced by the current group. The paper is organized as follows. In §2, inspired by [4], we define an extension $\hat{\mathbf{g}}(M)$ of the current algebra. In §3, we introduce a bilinear form on $\hat{\mathbf{g}}(M)$ which is shown to be invariant and nondegenerate. In §4, we prove a useful lemma. In §5, we study the adjoint action of the current group on $\hat{\mathbf{g}}(M)$ by automorphisms preserving the bilinear form. We find out that this can be done only when $\dim M \leq 2$. Finally, in §6, we show the previous results generalize the loop algebra case (formulas (2) to (4)) when we take $M = S^1$.

2. Extension of current algebras

Let G be a reductive Lie group and $\mathbf{g}$ its Lie algebra. There is a nondegenerate symmetric bilinear invariant form $(\cdot|\cdot)$ on $\mathbf{g}$. Suppose M is a compact, closed, orientable manifold with a normalized volume form Ω. Let $\tilde{\mathbf{g}}$ be the set of all C^∞ maps $x\colon M \to \mathbf{g}$. $\tilde{\mathbf{g}}$ is the Lie algebra of the infinite-dimensional Lie group $\widetilde{G}$ of all C^∞ maps $g\colon G \to M$. It is called the current algebra. We now construct a (noncentral) extension of $\tilde{\mathbf{g}}$:

$$\hat{\mathbf{g}} = \tilde{\mathbf{g}} \oplus \mathbf{c} \oplus \mathbf{d}, \tag{5}$$

where

$$\mathbf{c} = \Lambda^1(M)/d\Lambda^0(M), \tag{6}$$

and

$$\mathbf{d} = \{X \in \mathrm{Vect}(M) | L_X\Omega = 0\}, \tag{7}$$

where $L_X = di_X + i_X d$ is the Lie derivative operator.

The commutation relations are given by

$$[x, y]^\wedge = [x, y] + (dx|y) = [x, y] - (x|dy), \tag{8}$$

$$[\alpha, x]^\wedge = 0, \tag{9}$$

$$[X, x]^\wedge = L_X x, \tag{10}$$

$$[\alpha, \beta]^\wedge = 0, \tag{11}$$

$$[X, \alpha]^\wedge = L_X \alpha, \tag{12}$$

$$[X, Y]^\wedge = [X, Y], \text{ the usual bracket of vector fields}, \tag{13}$$

for any $x, y \in \tilde{\mathbf{g}}$, $\alpha, \beta \in \mathbf{c}$ and $X, Y \in \mathbf{d}$. The anticommutativity and the Jacobi identities follow easily from the properties of usual brackets and Lie derivatives.

A similar definition was given by Kač in [4], where $\hat{\mathbf{g}}$ is a central extension of $\tilde{\mathbf{g}}$. In our case, $[X, \alpha]^\wedge$ is not zero in order to define automorphisms of the current algebra. See remark of §6 for details.

3. The nondegenerate invariant form on $\hat{\mathbf{g}}$

We define a symmetric bilinear form on $\hat{\mathbf{g}}$ by the following formulas

$$(x|y)^\wedge = \int_M (x|y)\Omega, \tag{14}$$

$$(X|\alpha)^\wedge = \int_M (i_X\alpha)\Omega, \tag{15}$$

and the others are zero. It is well defined because for each $f \in C^\infty(M)$,

$$\begin{aligned}(X|df)^\wedge &= \int_M (i_X\, df)\Omega = \int_M (L_X f)\Omega \\ &= \int_M L_X(f\Omega) = \int_M d i_X(f\Omega) \\ &= \int_{\partial M} i_X(f\Omega) = 0.\end{aligned} \tag{16}$$

The nondegeneracy and invariance properties are given by the following two propositions.

Proposition 3.1. *The form* $(\cdot|\cdot)^\wedge$ *is nondegenerate.*

Proof. Notice that if $x \neq 0$, then there exists a $y \in \tilde{\mathbf{g}}$ such that $(x|y)^\wedge \neq 0$; also if $X \neq 0$, then there exists an $\alpha \in \mathbf{c}$ such that $(X|\alpha)^\wedge \neq 0$. So it suffices to check that $\alpha \in \Lambda^1(M)$ is exact if it satisfies $(X|\alpha)^\wedge = 0$ for all $X \in \mathbf{d}$. Since

$$(X|\alpha)^\wedge = \int_M (i_X\alpha)\Omega = \int_M \alpha \wedge i_X\Omega, \tag{17}$$

we only have to show that the map $\Theta\colon \mathbf{d} \to Z^{n-1}(M)$ defined by $\Theta(X) = i_X\Omega$ is onto (by Poincaré duality). It is easy to see that for each $\beta \in Z^{n-1}(M)$, there is an $X \in \mathrm{Vect}(M)$ such that $i_X\Omega = \beta$; $d\beta = 0$ implies that $L_X\Omega = di_X\Omega = 0$, which means $X \in \mathbf{d}$. □

Proposition 3.2. *The form* $(\cdot|\cdot)^\wedge$ *is invariant.*

Proof. By definition of invariance, we have to check that the left-hand sides of the following two formulas are equal:

$$\begin{aligned}&([x+\alpha+X, y+\beta+Y]^\wedge|z+\gamma+Z)^\wedge \\ &\quad= \int_M \{([x,y]|z) + (L_X y - L_Y x|z) + i_Z(dx|y) \\ &\qquad+ i_Z(L_X\beta - L_Y\alpha) + i_{[X,Y]}\gamma\}\Omega,\end{aligned} \tag{18}$$

$$\begin{aligned}&(x+\alpha+X|[y+\beta+Y, z+\gamma+Z]^\wedge)^\wedge \\ &\quad= \int_M \{(x|[y,z]) + (x|L_Y z - L_Z y) + i_X(dy|z) \\ &\qquad+ i_X(L_Y\gamma - L_Z\beta) + i_{[Y,Z]}\alpha\}\Omega.\end{aligned} \tag{19}$$

By using the well-known formula $i_{[X,Y]} = [i_X, L_Y] = [L_X, i_Y]$, the difference of the two integrands on the right-hand sides is a linear combination of Lie derivatives of functions on M

$$-L_Y(x|z) + L_Z(x|y) - L_Y(i_X\gamma) + L_Y(i_Z\alpha), \tag{20}$$

hence the integration over M is zero (see equation (16)). □

4. A LEMMA

We establish here a lemma which will be frequently used in §5.

Lemma. *If $g\colon M \to G$ and $x\colon M \to \mathbf{g}$ are C^∞-maps, $X \in \mathrm{Vect}(M)$, then*

$$\text{(a)} \quad L_X(\mathrm{Ad}\, g)_* x = (\mathrm{Ad}\, g)_*(L_X x + [i_X g^*\omega, x]), \tag{21}$$

$$\text{(b)} \quad d(\mathrm{Ad}\, g)_* x = (\mathrm{Ad}\, g)_*(dx + [dg^*\omega, x]), \tag{22}$$

where $(\mathrm{Ad}\, g)_*$ is the adjoint action of $g \in G$ on $\mathbf{g}$ and ω is the Maurer-Cartan form of the Lie group G.

Proof. Let ϕ_t be the flow on M generated by X. Set $g_t = g \circ \phi_t\colon M \to G, x_t = x \circ \phi_t\colon M \to \mathbf{g}$. Then

$$L_X(\mathrm{Ad}\, g)_* x = \lim_{t\to 0} \frac{(\mathrm{Ad}\, g_t)_* x_t - (\mathrm{Ad}\, g)_* x}{t} \tag{23}$$

$$= (\mathrm{Ad}\, g)_* \left(L_X x + \lim_{t\to 0} \frac{(\mathrm{Ad}\, g^{-1} g_t)_* - 1}{t} x \right). \tag{24}$$

Notice that the tangent vector of the flow $g^{-1} g_t$ at the unit element of G is $i_X g^*\omega$, an element in the Lie algebra $\mathbf{g}$. Hence we have

$$\lim_{t\to 0} \frac{(\mathrm{Ad}\, g^{-1} g_t)_* - 1}{t} x = [i_X g^*\omega, x]. \tag{25}$$

This proves part (a). Part (b) follows easily. □

5. AUTOMORPHISMS OF $\hat{\mathbf{g}}$

We now investigate how to define the adjoint action of $\widetilde{G}$ on the extended current algebra $\hat{\mathbf{g}}$ so that the bilinear form $(\cdot|\cdot)^\wedge$ is preserved. That is to say, for each $g \in \widetilde{G}, x \in \tilde{\mathbf{g}}$ and $X \in \mathbf{d}$ we have to find $a(g, X) \in \tilde{\mathbf{g}}$ and $\xi(g, x), \eta(g, X) \in \Lambda^1(M)$ such that the action $\widehat{\mathrm{Ad}} g$ on $\hat{\mathbf{g}}$ defined by

$$\begin{aligned} (\widehat{\mathrm{Ad}} g)x &= (\mathrm{Ad}\, g)_* x + \xi(g, x) \\ (\widehat{\mathrm{Ad}} g)\alpha &= \alpha \\ (\widehat{\mathrm{Ad}} g)X &= a(g, X) + \eta(g, X) + X \end{aligned} \tag{26}$$

is an automorphism which preserves $(\cdot|\cdot)^\wedge$; here $(\mathrm{Ad}\, g)_*$ is the pointwise adjoint action of $\widetilde{G}$ on $\tilde{\mathbf{g}}$.

Since we require

$$[(\widehat{\mathrm{Ad}}g)X, (\widehat{\mathrm{Ad}}g)x]^\wedge = (\widehat{\mathrm{Ad}}g)[X, x]^\wedge, \tag{27}$$

we must have, by comparing the $\tilde{\mathfrak{g}}$-components,

$$[a, (\mathrm{Ad}\, g)_* x] + L_X(\mathrm{Ad}\, g)_* x = (\mathrm{Ad}\, g)_* L_X x. \tag{28}$$

We can choose

$$a(g, X) = -(\mathrm{Ad}\, g)_* i_X g^* \omega \tag{29}$$

by the lemma, where ω is the Maurer–Cartan form of G. Next, invariance implies

$$((\mathrm{Ad}\, g)_* x, (\mathrm{Ad}\, g)_* X)^\wedge = (x, X)^\wedge = 0. \tag{30}$$

By the choice of $a(g, X)$, we have

$$(\xi|X)^\wedge = ((\mathrm{Ad}\, g)_* x|(\mathrm{Ad}\, g)_* i_X g^* \omega)^\wedge, \tag{31}$$

i.e.

$$\begin{aligned}\int_M (i_X \xi)\Omega &= \int_M ((\mathrm{Ad}\, g)_* x|(\mathrm{Ad}\, g)_* i_X g^* \omega)\Omega \\ &= \int_M (i_X(x|g^*\omega))\Omega.\end{aligned} \tag{32}$$

This means we can choose

$$\xi(g, x) = (x|g^*\omega). \tag{33}$$

Finally,

$$((\widehat{\mathrm{Ad}}g)X|(\widehat{\mathrm{Ad}}g)X)^\wedge = (X|X)^\wedge = 0 \tag{34}$$

implies

$$(\eta|X)^\wedge = -\tfrac{1}{2}((\mathrm{Ad}\, g)_* i_X g^* \omega|(\mathrm{Ad}\, g)_* i_X g^* \omega)^\wedge. \tag{35}$$

Proceeding as in (32), we can choose

$$\eta(g, X) = -\tfrac{1}{2}(i_X g^* \omega|g^*\omega). \tag{36}$$

We summarize in the following.

Theorem. *For each $g \in \widetilde{G}$, the action $\widehat{\mathrm{Ad}}g$ given by*

$$\begin{aligned}(\widehat{\mathrm{Ad}}g)x &= (\mathrm{Ad}\, g)_* x + (x|g^*\omega) \\ (\widehat{\mathrm{Ad}}g)\alpha &= \alpha \\ (\widehat{\mathrm{Ad}}g)X &= -(\mathrm{Ad}\, g)_* i_X g^* \omega - \tfrac{1}{2}(i_X g^*\omega|g^*\omega) + X\end{aligned} \tag{37}$$

preserves the bilinear form $(\cdot|\cdot)^\wedge$. Furthermore, it is an automorphism of $\hat{\mathfrak{g}}$ if $\dim M \le 2$.

In order to keep the proof to a reasonable size, routine calculations will be omitted.

Proof. From the construction of $\widehat{\mathrm{Ad}}g$, it is easy to see that it preserves the bilinear form $(\cdot|\cdot)^\wedge$. That $\widehat{\mathrm{Ad}}g$ preserves the bracket of x and α, X and α, α and β is also straightforward. Next, we calculate

$$\begin{aligned}
&[(\widehat{\mathrm{Ad}}g)x, (\widehat{\mathrm{Ad}}g)y]^\wedge \\
&\quad = [(\mathrm{Ad}\, g)_* x, (\mathrm{Ad}\, g)_* y] + (d(\mathrm{Ad}\, g)_* x|(\mathrm{Ad}\, g)_* y) \\
&\quad = (\mathrm{Ad}\, g)_*[x, y] + ((\mathrm{Ad}\, g)_*(dx + [g^*\omega, x])|(\mathrm{Ad}\, g)_* y) \quad \text{(by the lemma)} \\
&\quad = (\mathrm{Ad}\, g)_*[x, y] + (g^*\omega|[x, y]) + (dx|y) \\
&\quad = (\widehat{\mathrm{Ad}}g)[x, y]^\wedge
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&[(\widehat{\mathrm{Ad}}g)X, (\widehat{\mathrm{Ad}}g)x]^\wedge - (\widehat{\mathrm{Ad}}g)[X, x]^\wedge \\
&\quad = (i_X g^*\omega|dx + [g^*\omega, x]) + (x|L_X g^*\omega) \quad \text{(by the lemma)} \\
&\quad = d(i_X g^*\omega|x) + (i_X d g^*\omega|x) + ([i_X g^*\omega, g^*\omega]|x) \\
&\quad = d(i_X g^*\omega|x) + i_X(g^*(d\omega + \tfrac{1}{2}[\omega, \omega])|x) \\
&\quad = d(i_X g^*\omega|x),
\end{aligned} \tag{39}$$

which is an exact form. The last step above follows from the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$. We finally examine the difference of $[(\widehat{\mathrm{Ad}}g)X, (\widehat{\mathrm{Ad}}g)Y]^\wedge$ and $(\widehat{\mathrm{Ad}}g)[X, Y]^\wedge$. Their $\mathbf{d}$-components are both $[X, Y]$. The difference of the $\tilde{\mathbf{g}}$-components is

$$\begin{aligned}
&(\mathrm{Ad}\, g)_*[i_X g^*\omega, i_Y g^*\omega] + L_Y(\mathrm{Ad}\, g)_* i_X g^*\omega - L_X(\mathrm{Ad}\, g)_* i_Y g^*\omega \\
&\qquad + (\mathrm{Ad}\, g)_* i_{[X,Y]} g^*\omega \\
&\qquad = (\mathrm{Ad}\, g)_*\{(L_Y i_X - i_Y L_X) g^*\omega - [i_X g^*\omega, i_Y g^*\omega]\} \\
&\qquad = (\mathrm{Ad}\, g)_*\{-i_X i_Y g^*(d\omega + \tfrac{1}{2}[\omega, \omega])\} \\
&\qquad = 0.
\end{aligned} \tag{40}$$

The difference of the $\mathbf{c}$-components is

$$\begin{aligned}
&\tfrac{1}{2}(L_Y i_X g^*\omega|g^*\omega) + \tfrac{1}{2}(i_X g^*\omega|L_Y g^*\omega) - \tfrac{1}{2}(L_X i_Y g^*\omega|g^*\omega) - \tfrac{1}{2}(i_Y g^*\omega|L_X g^*\omega) \\
&\qquad + (d i_X g^*\omega + [g^*\omega, i_X g^*\omega]|i_Y g^*\omega) + \tfrac{1}{2}(i_{[X,Y]} g^*\omega|g^*\omega) \\
&\quad = \tfrac{1}{2}((L_Y i_X - L_X i_Y + [L_X, i_Y]) g^*\omega|g^*\omega) + ([i_X g^*\omega, i_Y g^*\omega]|g^*\omega) \\
&\qquad + \tfrac{1}{2}(i_X g^*\omega|(d i_Y + i_Y d) g^*\omega) + \tfrac{1}{2}((d i_X - i_X d) g^*\omega|i_Y g^*\omega) \\
&\quad = \tfrac{1}{2} d(i_X g^*\omega|i_Y g^*\omega) + \tfrac{1}{2}((d i_Y i_X - i_Y i_X d) g^*\omega|g^*\omega) \\
&\qquad + ([i_X g^*\omega, i_Y g^*\omega]|g^*\omega) \\
&\qquad + \tfrac{1}{2}(i_X g^*\omega| - \tfrac{1}{2} i_Y [g^*\omega, g^*\omega]) + \tfrac{1}{2}(\tfrac{1}{2} i_X [g^*\omega, g^*\omega]|i_Y g^*\omega) \\
&\quad = \tfrac{1}{2} d(i_X g^*\omega|i_Y g^*\omega) + \tfrac{1}{2}([i_X g^*\omega, i_Y g^*\omega]|g^*\omega).
\end{aligned} \tag{41}$$

The first term above is an exact form, but the second one is only when $\dim M \le 2$. To see this, we introduce local coordinates (σ, τ) when $\dim M = 2$ and choose $X = \partial/\partial\sigma$ and $Y = \partial/\partial\tau$. Then

$$\begin{aligned}(42)\quad &([i_X g^*\omega, i_Y g^*\omega]|g^*\omega) \\ &= ([i_{\partial/\partial\sigma} g^*\omega, i_{\partial/\partial\tau} g^*\omega]|i_{\partial/\partial\sigma} g^*\omega)\, d\sigma \\ &\quad + ([i_{\partial/\partial\sigma} g^*\omega, i_{\partial/\partial\tau} g^*\omega]|i_{\partial/\partial\tau} g^*\omega)\, d\tau = 0,\end{aligned}$$

since the bilinear form is invariant. □

Remark. If $\dim M \geq 3$, the second term in (41) is not necessarily exact. Take $M = T^3$ parametrized by (θ, φ, ψ) and $G = SU(2)$. Choose $X = \partial/\partial\theta$. $Y = \partial/\partial\varphi$ and

$$g = \begin{pmatrix} e^{i\theta}\cos\varphi & e^{i\psi}\sin\varphi \\ -e^{i\psi}\sin\varphi & e^{-i\theta}\cos\varphi \end{pmatrix}. \tag{43}$$

Then

$$\tfrac{1}{2}([i_X g^*\omega, i_Y g^*\omega]|g^*\omega) = -2\sin 2\varphi\, d\psi, \tag{44}$$

which is not even closed.

6. Comparisons with Kač-Moody algebras

We now take $M = S^1$ parametrized by $e^{i\theta}$. Its normalized volume form is $\Omega = d\theta/2\pi$. In this case, $\tilde{\mathbf{g}}$ is the loop algebra. $\mathbf{c} = H^1(S^1, \mathbf{C})$ is a one-dimensional vector space generated by the element $c = i\,d\theta$; every one form α is represented by $((1/2\pi i)\int_{S^1}\alpha)c$. $\mathbf{d}$ is also one dimensional, generated by $d = (1/i)(d/d\theta)$. Then the commutation relations (8) to (13) become

$$\begin{aligned}&[x, y]^\wedge = [x, y] + \frac{1}{2\pi i}\int_{S^1} x'(\theta)y(\theta)\, d\theta\, c \\ &[x, c]^\wedge = 0 \\ &[d, x]^\wedge = \frac{1}{i}\frac{d}{d\theta}x \\ &[c, d]^\wedge = 0.\end{aligned} \tag{45}$$

The invariant form on $\hat{\mathbf{g}}$ is given by

$$\begin{aligned}&(x|y)^\wedge = \frac{1}{2\pi}\int_{S^1}(x(\theta)|y(\theta))\, d\theta \\ &(c|d)^\wedge = (d|c)^\wedge = 1 \\ &\text{others are zero.}\end{aligned} \tag{46}$$

Finally, by the Theorem the action of $\widetilde{G}$ on $\hat{\mathbf{g}}$ is

$$\begin{aligned}&(\widehat{\mathrm{Ad}}g)x = gxg^{-1} + \frac{1}{2\pi i}\int_{S^1}\mathrm{tr}(xg^{-1}dg)c \\ &(\widehat{\mathrm{Ad}}g)c = c \\ &(\widehat{\mathrm{Ad}}g)d = ig'g^{-1} - \frac{1}{4\pi}\int_{S^1}\mathrm{tr}((g^{-1})'dg)c + d.\end{aligned} \tag{47}$$

All these formulas agree with those given in the introduction if we make a change of variable $t = e^{i\theta}$ there.

Remark. If M is one dimensional, then $\mathbf{c}$ is in the center of $\hat{\mathbf{g}}$ because $[X, \alpha]^\wedge = L_X\alpha$ is exact. This is not the case in general. If we had defined $[X, \alpha]^\wedge = 0$ instead of $L_X\alpha$, then the $\mathbf{c}$-component of $[(\widehat{\mathrm{Ad}}g)X, (\widehat{\mathrm{Ad}}g)x]^\wedge - (\widehat{\mathrm{Ad}}g)[X, x]^\wedge$ would be $-L_X(x|g^*\omega)$; this may not be exact.

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