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ALGEBRAIC STRUCTURE IN COMPLEX FUNCTION SPACES

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ABSTRACT. Let M be a complex function space containing constants, and let Z be the complex state space of M. If M is linearly isometric to a uniform algebra and if Z is affinely homeomorphic to the complex state space of a uniform algebra then we prove that M is a uniform algebra. Neither of the two conditions taken separately imply this conclusion.

If X is a compact Hausdorff space then $C_{\mathbf{C}}(X)$, $C_{\mathbf{R}}(X)$ will denote the Banach spaces of all continuous complex-valued, respectively real-valued, functions on X with the supremum norm. A closed linear subspace M of $C_{\mathbf{C}}(X)$ which contains constants and separates the points of X will be called a *complex function space*. The subset $S = \{\varphi \in M^* : \|\varphi\| = 1 = \varphi(1)\}$ of M^* is called the *state space* of M and the subset $Z = \mathrm{co}(S \cup -iS)$ of M^* is called the *complex state space* of M; the sets S and Z are compact convex sets when endowed with the relative w^*-topology. If K is any compact convex subset of a locally convex Hausdorff space then $A(K)$, $A_{\mathbf{C}}(K)$ will denote the Banach spaces of all continuous real-valued, respectively complex-valued, affine functions on K with the supremum norm.

We shall be concerned with the linear and norm structure of M and the affine and topological structure of Z and will seek conditions which imply that M is a uniform algebra on X. To this end we say that the complex state spaces Z_1, Z_2 of two complex function spaces M_1, M_2 are *equivalent* if they are affinely homeomorphic, and that Z_1, Z_2 are *real-equivalent* if there is an affine homeomorphism $\eta : Z_1 \to Z_2$ which maps S_1 onto S_2 (and hence maps $-iS_1$ onto $-iS_2$).

We begin by developing [4, Examples 3 and 1] to show that the property that M has the linear and norm structure of a uniform algebra is independent from the property that Z is equivalent (or real-equivalent) to the complex state space of a uniform algebra.

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Example 1. Let $M_1 = P(\Gamma)$ be the disc algebra on the unit circle Γ, and let $M = \{\bar{z}f(z) : f \in M_1\}$. Then [4, Example 3] shows that M and M_1 are isometrically isomorphic while Z and Z_1 are not equivalent. We will show that Z is not equivalent to Z_2 for any uniform algebra M_2.

Suppose that Z is equivalent to Z_2. Then the connectedness of Γ implies that either Z is real-equivalent to Z_2 or Z is real-equivalent to the complex state space of $\bar{M}_2$ (cf. [5]). We may hence assume that Z and Z_2 are real-equivalent, and that M_2 is a uniform algebra on Γ (cf. [4]). We have $M_2 = \{u + iv \circ \psi : u + iv \in M\}$, where $\psi : \Gamma \to \Gamma$ is a homeomorphism.

Now z and $\bar{z}$ belong to M and so the functions $f(z) = x + \operatorname{im} \psi(z)$ and $\bar{f}(z) = x - \operatorname{im} \psi(z)$ belong to M_2, where we write $z = x + iy$. Hence the function $g(z) = x$ belongs to M_2 and similarly, using the facts that iz and $i\bar{z}$ belong to M, we see that the function $h(z) = y$ belongs to M_2. Consequently the uniform algebra M_2 equals $C_{\mathbf{C}}(\Gamma)$. This implies that $M = C_{\mathbf{C}}(\Gamma)$, giving the required contradiction.

Example 2. Let $M_1 = P(\Gamma)$ and $M = \{f : f(z) = u(z) + iv(-z)$ for some $u + iv \in M_1\}$. Then [4, Example 1] shows that Z and Z_1 are real-equivalent while M and M_1 are not isometrically isomorphic. We will show that M is not isometrically isomorphic to any uniform algebra M_2.

If M is isometrically isomorphic to M_2, a uniform algebra on X, then we will have $M_2 = \{\lambda(f \circ \tau) : f \in M\}$, where $\lambda \in M_2$ with $|\lambda| = 1$ and $\tau : X \to \Gamma$ a homeomorphism. Writing $M_3 = \{g \circ \tau^{-1} : g \in M_2\}$ we note that M_3 is a uniform algebra on Γ equal to $\{(\lambda \circ \tau^{-1})f : f \in M\} = \{lf : f \in M\}$, where $l = \lambda \circ \tau^{-1} \in M_3$ with $|l| = 1$. Since z^{2n} and $\bar{z}^{2n+1}$ belong to M, $n \geq 0$, the functions $l(z)z^{2n}$ and $l(z)\bar{z}^{2n+1}$ belong to M_3. Since M_3 is an algebra and since $l(z)^2 z^{2n} = l(z)(l(z)z^{2n})$, $l(z)^2 \bar{z}^{2n+1} = l(z)(l(z)\bar{z}^{2n+1})$, $l(z)^2 z^{2n+1} = (l(z)\bar{z})(l(z)z^{2n+2})$, $l(z)^2 \bar{z}^{2n} = (l(z)\bar{z})(l(z)\bar{z}^{2n+1})$ we see that the functions $l(z)^2 z^k$, k any integer, belong to M_3. Since the polynomials in z and $\bar{z}$ form a dense linear subspace of $C_{\mathbf{C}}(\Gamma)$ it follows that $M_3 = C_{\mathbf{C}}(\Gamma)$. This implies that $M = C_{\mathbf{C}}(\Gamma)$, giving the required contradiction.

We will now show that if M has the linear and norm structure of a uniform algebra, and if Z is equivalent to the complex state space of a uniform algebra, then M is necessarily a uniform algebra. We note firstly however that we cannot replace 'complex state space' by 'state space' in this result. Indeed, in Example 1 above M and M_1 are isometrically isomorphic and, since M contains the Dirichlet algebra M_1, the state spaces of M and M_2 are equivalent to the state space of $C_{\mathbf{R}}(\Gamma)$.

We need to recall some concepts, full details of which may be found in Asimow and Ellis [1]. The *centre* of $A(K)$ consists of those functions $f \in A(K)$ such that for each $G \in A(K)$ there is some $h \in A(K)$ satisfying $h(x) = f(x)g(x)$ for all $x \in \partial K$, where ∂K denotes the set of extreme points of K. The sets of constancy in ∂K for the central functions in $A(K)$ form the sets of extreme points of a family of faces $\{F_\alpha\}$ of K, called the *Šilov decomposition* for $A(K)$. The maximal subsets E of ∂K such that the centre of $A(\overline{\text{co}}E)$ is trivial form the sets of extreme points of a family of faces $\{F_\beta\}$ of K called the *Bishop decomposition* for $A(K)$. In the case when K is the complex state space of a uniform algebra these decompositions are closely related to the corresponding classical decompositions.

If Z is the complex state space of a function space M then $\theta : M \to A(Z)$ will denote the real-linear homeomorphism defined by $\theta f(z) = \operatorname{re} z(f)$, noting that $\theta(u+iv)(\lambda x - i(1-\lambda)y) = \lambda u(x) + (1-\lambda)v(y)$ when $x, y \in X$ and $0 \le \lambda \le 1$. For this purpose we consider X to be canonically embedded in S. θ_1, θ_2 will denote the corresponding maps for M_1 and M_2.

Theorem 1. *Let M be a complex function space on X with complex state space Z, and let M_j be uniform algebras with complex state spaces Z_j, $j = 1, 2$. If M is isometrically isomorphic to M_1 and if Z is equivalent to Z_2 then M is a uniform algebra on X.*

Proof. We first prove the result in the special case when Z is real-equivalent to Z_2.

As in the discussion of the Examples above we may assume that M_1, M_2 are uniform algebras on X, and that

$$M = \{lf : f \in M_1\} = \{u + iv \circ \psi : u + iv \in M_2\},$$

where $l \in M$ with $|l| = 1$ and $\psi : X \to X$ is a homeomorphism with ψ^2 equal to the identity map on the essential set for M_2. In order to prove that M is an algebra it will be sufficient to show that $l \in M_1$, that is $l^2 \in M$. Write $l = g + ih$ so that $g + ih = u + iv \circ \psi$ for some $u + iv \in M_2$. Since M contains constants we must have $\bar{l} \in M_1$, so that $\bar{l} = l\bar{l}^2 \in M$. Hence $g - ih = u_1 + iv_1 \circ \psi$ for some $u_1 + iv_1 \in M_2$. Consequently we obtain $g = u = u_1$, $h = v \circ \psi = -v_1 \circ \psi$, so that $v = -v_1$ and $u - iv$, u and v belong to M_2. But then $l^2 = g^2 - h^2 + 2igh = 2u^2 - 1 + 2i((u \circ \psi^{-1})v) \circ \psi$ belongs to M because $u \circ \psi^{-1}$ belongs to M_2 (cf. [4]). Hence M is an algebra.

We now turn to the general case where Z and Z_2 are equivalent and $M = \{lf : f \in M_1\}$, with $l \in M$. Firstly we identify the centres of $A(Z)$ and $A(Z_1)$. The centre of $A(Z_1)$ consists of the functions $\theta_1(u + iv)$ such that u, v belong to M_1 and $u - v$ belongs to the essential ideal for M_1 (cf. [2, Theorem 1]).

Suppose that $\theta(u+iv)$ belongs to the centre of $A(Z)$. Then for each $a+ib \in M$ we have $ua + ivb$ belongs to M, and since 1, i belong to M we may deduce that u and v belong to M. If $a+ib \in M$ then we have $b-ia \in M$ and hence $ub-iva$ and $va+iub$ belong to M. Consequently $ua+ivb+va+iub = (u+v)(a+ib)$ belongs to M and $ua+ivb-va-iub = (u-v)(a-ib)$ belongs to M. Conversely, reversing this argument, we see that $\theta(u+iv)$ belongs to the centre of $A(Z)$ whenever $(u+v)M$ and $(u-v)\overline{M}$ are contained in M.

Therefore $\theta(u+iv)$ belongs to the centre of $A(Z)$ if and only if $f \in M_1$ implies that $(u+v)lf$ and $(u-v)\bar{l}\bar{f}$ belong to M, that is $(u+v)f$ and $(u-v)\bar{l}^2\bar{f}$ belong to M_1. Taking $f = 1$ and also $f = \bar{l}^2 \in M_1$, we see that $u+v$, $u-v$, u and v belong to M_1 whenever $\theta(u+iv)$ belongs to the centre of $A(Z)$. In this case taking $f = \bar{l}^2 g$, where $g \in M_1$, we see that $(u-v)\bar{g} \in M_1$; since $(u-v)g$ belongs to M_1 it follows that $(u-v)\operatorname{re} g$ and $(u-v)\operatorname{im} g$ belong to M_1. The proof of [2, Theorem 1] now shows that $u-v$ belongs to the essential ideal I_1 of M_1, that is $\theta_1(u+iv)$ belongs to the centre of $A(Z_1)$. Conversely, if u, $v \in M_1$, and $u-v \in I_1$ then, for all $f \in M_1$, we have $(u+v)f$, $(u-v)\bar{l}^2\bar{f} \in M_1$, because $\bar{l}^2\bar{f} \in C_{\mathbf{C}}(X)$. Hence $\theta(u+iv)$ belongs to the centre of $A(Z)$, and we have shown that the centres of $A(Z)$ and $A(Z_1)$ may be identified.

We may assume without loss of generality that X is the Šilov boundary for both M_1 and M. Therefore we have shown that the Šilov decompositions of $Y = X \cup -iX$ corresponding to $A(Z)$ and $A(Z_1)$ coincide. The Šilov decompositions of Y for $A(Z_1)$, except for the singleton sets, consists of sets of the form $E_\alpha \cup -iE_\alpha$, where E_α belongs to the Šilov decomposition of X for M_1 (cf. [3]). Now $M_1|E_\alpha$ and $M|E_\alpha$ are isometrically isomorphic, and we may apply the preceding reasoning to these spaces to conclude that the Bishop decompositions of Y corresponding to $A(Z)$ and $A(Z_1)$ coincide.

The Bishop decomposition for $A(Z)$, except for singletons, consists of faces of the form $G_\beta = \operatorname{co}(F_\beta \cup -iF_\beta)$, where $F_\beta \cap X = E_\beta$ belongs to the Bishop decomposition for M_1. Moreover, if $g \in C_{\mathbf{C}}(X)$ is such that $g|E_\beta$ belongs to $M|E_\beta$ for all β, then $\bar{l}g \in C_{\mathbf{C}}(X)$ and $\bar{l}g|E_\beta \in M_1|E_\beta$ for all β which implies that $\bar{l}g \in M_1$, and hence g belongs to M. We can hence conclude that M is an algebra if we can show that $l^2|E_\beta$ belongs to $M|E_\beta$ for all β.

Since Z is equivalent to Z_2 the faces of the Bishop decompositions for Z and Z_2 are equivalent. Therefore if we restrict attention to $M|E_\beta$ and $M_1|E_\beta$ we see that the complex state space G_β of $M|E_\beta$ is equivalent to the complex state space of an antisymmetric uniform algebra M_3 (a restriction algebra of M_2). However in this case either G_β is real-equivalent to Z_3 or is real-equivalent to the complex state space of $\overline{M}^3$. In either case the first part of the proof shows that $M|E_\beta$ is an algebra. Consequently $l^2|E_\beta$ belongs to $M|E_\beta$ and the proof of the theorem is complete.

We remark that the condition in Theorem 1 that Z is equivalent to Z_2 is much weaker than the condition that Z is real-equivalent to Z_2. In fact if M is self-adjoint and if Z is real-equivalent to Z_2 then, since S is a split face of Z, S_2 must be a split face of Z_2 which implies that M_2 is a $C_{\mathbf{C}}(X)$-space. This conclusion need not hold when Z and Z_2 are just equivalent, as the following example shows.

Example 3. Let Z_Γ, $Z_{\Gamma'}$ denote respectively the complex state spaces of $P(\Gamma)$, $P(\Gamma')$, where $P(\Gamma')$ is the uniform algebra generated by the polynomials on

$\Gamma' = \{z \in \mathbf{C} : |z-3| = 1\}$. Let $M = A_{\mathbf{C}}(Z_\Gamma)$ and $M_2 = P(\Gamma \cup \Gamma')$. Then Z_2 is the convex hull of the disjoint closed split faces Z_Γ and $Z_{\Gamma'}$, while Z is the convex hull of the disjoint closed split faces Z_Γ and $-iZ_\Gamma$. Since $-iZ_\Gamma$ and $Z_{\Gamma'}$ are equivalent so are Z and Z_2. In this example M is self-adjoint while the uniform algebra M_2 is not a $C_{\mathbf{C}}(X)$-space.

We note that it is easy to verify that no non-trivial uniform algebra can be isometrically isomorphic to a self-adjoint complex function space.

In the context of Theorem 1, Nagasawa's theorem [7] show that M_1 is unique in the sense that any two isometrically isomorphic uniform algebras are algebraically isomorphic. On the other hand M_2 need not be unique even if M is a $C_{\mathbf{C}}(X)$-space (cf. [4, Example 2]). Our final result gives conditions under which M_2 is uniquely determined, up to complex conjugation. A related result appeared in Ellis and So [5, Corollary 6]).

Theorem 2. *Let M_1, M_2 be uniform algebras with essential sets X, Y respectively. If Z_1 and Z_2 are equivalent then $M_2|Y$ is isometrically isomorphic to $(M_1|E) \otimes \bar{M}_1|X\backslash E)$, for some open and closed subset E of X.*

Proof. Let $\varphi : Z_1 \to Z_2$ be an equivalence. The essential face for Z_1 has the form $\operatorname{co}(F \cup -iF)$, where F is the closed convex hull of X in S_1 (cf. [3, Proposition 17]). Since φ maps the essential face of Z_1 onto the essential face of Z_2, and since $\operatorname{co}(F \cup -iF)$ is the complex state space of $M_1|X$, we can assume without loss of generality that M_1 and M_2 are essential uniform algebras, and that X, Y are the Šilov boundaries of M_1, M_2 respectively.

If we write $E = \{x \in X : \varphi(x) \in S_2\}$ then $X = E \cup (X\backslash E)$ is a peak-set decomposition of X for M_1 (cf. [5, Corollary 2]). Since M_1 is essential so are the algebras $M_1|E$ and $M_1|(X\backslash E)$ and hence E (respectively $X\backslash E$) is the closure of the union of non-singleton maximal antisymmetric sets for $M_1|E$ (respectively $M_1|(X\backslash E)$) (cf. [6, page 65]).

The faces of the Bishop decomposition for Z_1 are the singletons x, $-ix$, where x is a singleton member of the Bishop decomposition for M_1, together with faces of the form $\operatorname{co}(F_\alpha \cup -iF_\alpha)$, where F_α is the closed convex hull in S_1 of a non-singleton member of the Bishop decomposition for M_1. Now each $\operatorname{co}(F_\alpha \cup -iF_\alpha)$ is mapped by φ onto a corresponding member $\operatorname{co}(G_\alpha \cup -iG_\alpha)$ of the Bishop decomposition for Z_2. Consequently $\overline{\operatorname{co}}(E \cup -iE)$ is mapped onto a face of the form $\operatorname{co}(H \cup -iH)$, and similarly for $\overline{\operatorname{co}}((X\backslash E) \cup -i(X\backslash E))$. Since the Bishop decompositions determine M_1 and M_2, and since we have $M_1|(F_\alpha \cap X) = \{f \circ \varphi : f \in M_2|(G_\alpha \cap Y)\}$ whenever $F_\alpha \cap E$ is non-empty we see that $M_1|E = \{f \circ \varphi : f \in M_2|\varphi(E)\}$. Therefore $M_1|E$ is isometrically isomorphic to $M_2|\varphi(E)$. Similarly we may prove that $\bar{M}_1|(X\backslash E)$ is isometrically isomorphic to $M_2|(Y\backslash \varphi(E))$.

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