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# CONVERGENCE TO ENDS FOR RANDOM WALKS ON THE AUTOMORPHISM GROUP OF A TREE

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ABSTRACT. Let $\mu$ be a probability on a free group $\Gamma$ of rank $r \geq 2$. Assume that $\mathrm{Supp}(\mu)$ is not contained in a cyclic subgroup of $\Gamma$. We show that if $(X_n)_{n\geq 0}$ is the right random walk on $\Gamma$ determined by $\mu$, then with probability 1, $X_n$ converges (in the natural sense) to an infinite reduced word. The space $\Omega$ of infinite reduced words carries a unique probability $\nu$ such that $(\Omega, \nu)$ is a frontier of $(\Gamma, \mu)$ in the sense of Furstenberg [10]. This result extends to the right random walk $(X_n)$ determined by a probability $\mu$ on the group $G$ of automorphisms of an arbitrary infinite locally finite tree $T$. Assuming that $\mathrm{Supp}(\mu)$ is not contained in any amenable closed subgroup of $G$, then with probability 1 there is an *end* $\omega$ of $T$ such that $X_n v$ converges to $\omega$ for each $v \in T$. Our methods are principally drawn from [9] and [10].

## 1. INTRODUCTION AND PRELIMINARIES

Let $\mu$ be a probability on the free group $\Gamma$ generated by $\{a_1, \ldots, a_r\}$, where $r \geq 2$. Consider the right random walk $(X_n)_{n\geq 0}$ on $\Gamma$ determined by $\mu$; i.e., $X_0 = e$, and $X_n = Y_1 Y_2 \cdots Y_n$ for $n \geq 1$, where the $Y_j$'s are independent $\Gamma$-valued random variables on some probability space $(\Xi, \mathscr{F}, P)$, with $P(Y_j = x) = \mu(x)$ for each $x \in \Gamma$ and $j \geq 1$. Thus $(X_n)_{n\geq 0}$ is a Markov chain with transition probabilities $p(x, y) = \mu(x^{-1}y)$. It is well known that this chain is transient provided that it is irreducible, i.e., provided that for each $x \in \Gamma$ there is an $n \geq 1$ so that $\mu^{*n}(x) = P(X_n = x | X_0 = e) > 0$. In fact (see [1, p. 43] or [7]), it is transient provided that:

(1.1) The support $\mathrm{Supp}(\mu)$ of $\mu$ is not contained in a cyclic subgroup of $\Gamma$.

Indeed, every noncyclic subgroup of $\Gamma$ is nonamenable, by Schreier's Theorem [14]. Each $x \in \Gamma$ can be written in a unique way as a reduced word $x = x_1 x_2 \cdots x_m$, where $m \geq 0$, each $x_i$ is an $a_j$ or $a_j^{-1}$, and $x_{i+1} \neq x_i^{-1}$ for all $i < m$; we write $|x| = m$. Transience of $(X_n)$ just means that $|X_n| \to \infty$ with

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probability 1. A finer description of this transience is obtained by considering the space $\Omega$ of infinite reduced words $\omega = x_1 x_2 \cdots x_m \cdots$, where each $x_i$ is an $a_j$ or $a_j^{-1}$ and $x_{i+1} \neq x_i^{-1}$ for all $i$. It will follow from our main result (see Corollary 2.3, page 821) that assuming (1.1), with probability 1, $X_n$ converges to an $\omega$, i.e., for each $r \geq 1$ there is an $n_r$ so that once $n \geq n_r$, the first $r$ letters of the reduced word expression of $X_n$ agree with the first $r$ letters of $\omega$ (here $\omega$ depends on the point $\xi$ of the underlying probability space). This result has been proved under more restrictive conditions on $\mu$ in [6] (where $\mathrm{Supp}(\mu)$ is finite), [13] (where $\sum_{x\in\Gamma} \log(|x|+1)\mu(x) < \infty$), [5] ( where $\mu(x)$ depends only on $|x|$), and it is attributed (without proof) in [11] to G. A. Margulis (assuming only irreducibility?).

It follows from our result that $\Omega$, together with a measure $\nu$ it carries, uniquely determined by $\mu$, is a frontier for $(\Gamma, \mu)$ in the sense of [10].

Now let $T$ be an infinite locally finite tree (see, e.g., [14]). Let $G$ denote the group of *automorphisms* of $T$, i.e., mappings of $T$ onto itself which preserve the natural distance $d$ on $T$. Equipped with the topology of pointwise convergence, $G$ is a totally disconnected locally compact group. The subgroups $G(F) = \{\varphi \in G\colon \varphi(v) = v \text{ for all } v \in F\}$, where $F \subset T$ is finite, form a base of compact open neighbourhoods of the identity in $G$ (see, e.g., [15]). As in [4], the *ends* of $T$ are equivalence classes $\omega$ of infinite *chains*, i.e., sequences $(v_0, v_1, \dots)$ of distinct vertices of $T$, in which $v_i$ is a neighbour of $v_{i+1}$ for each $i$. Chains $(v_0, v_1, \dots)$ and $(v_0', v_1', \dots)$ are equivalent if there exist $r$ and $s$ so that $v_{r+i} = v_{s+i}'$ for all $i \geq 0$. If we fix a vertex $v_0$ in $T$, then each end $\omega$ contains exactly one chain starting from $v_0$, which we call the *geodesic* from $v_0$ to $\omega$. Let $\Omega$ denote the set of ends. There is a natural topology on $T \cup \Omega$, making it a compact Hausdorff space containing $T$ as a discrete dense subset. We may describe this topology as follows. Fix $v_0 \in T$. If $\omega \neq \omega' \in \Omega$, let $(v_0, v_1, \dots)$ and $(v_0' = v_0, v_1', \dots)$ be the geodesics from $v_0$ to $\omega$ and $\omega'$ respectively. Let $j(\omega, \omega')$ be the largest $j \geq 0$ such that $v_j = v_j'$, and set $j(\omega, \omega) = \infty$. We may similarly define $j(v, \omega)$ and $j(v, v')$ for $v$, $v' \in T$ and $\omega \in \Omega$. Then the sets $N_k = \{\omega \in T \cup \Omega\colon j(\omega, \omega_0) \geq k\}$, $k = 1, 2, \dots$, form a base of neighbourhoods of $\omega_0 \in \Omega$. Notice that $j(\omega, \omega')$ depends on $v_0$, but the topology does not. If $\omega \in \Omega$ and $\varphi \in G$, then $\varphi\omega$ is defined to be the end of the chain $(\varphi(v_0), \varphi(v_1), \dots)$, where $(v_0, v_1, \dots)$ is the geodesic from $v_0$ to $\omega$. This defines a continuous action of $G$ on $\Omega$.

Now let $\mu$ denote a probability measure on $G$ and let $Y_1$, $Y_2, \dots, Y_n, \dots$ denote $G$-valued i.i.d. random variables with common distribution $\mu$. Thus $X_0 = e$, $X_n = Y_1 Y_2 \cdots Y_n$ is a right random walk on $G$ with transition probabilities $p(x, B) = \mu(x^{-1}B)$ for all $x \in G$ and all Borel sets $B \subseteq G$. Since $T$ is a $G$-space (in the sense of Furstenberg) we may consider the process $X_n(v_0)$ where $v_0$ is any fixed element of $T$.

The main result of this paper consists in proving that if the support of $\mu$ is not contained in any amenable subgroup of $G$ then, with probability 1, there exists an end $\omega \in \Omega$ such that $X_n(v_0) \to \omega$ and $n \to \infty$ for every $v_0 \in T$.

The free group $\Gamma$ becomes a homogeneous tree of degree $2r$ if for each $x \in \Gamma$ an edge is drawn between $x$ and the $2r$ elements $xa_j^{\pm 1}$ (the Cayley graph). Left multiplication then provides a natural embedding of $\Gamma$ as a discrete subgroup of the group of automorphisms of this tree. Taking $v_0 = e$ we may identify ends with infinite reduced words. When $\varphi = g \in \Gamma$, $\varphi\omega$ is then obtained by multiplying the infinite reduced word $\omega$ on the left by $g$ and cancelling where possible.

## 2. Statements and proofs of results

Let $v_0$ be a fixed vertex of $T$, and write $|v|$ for $d(v_0, v)$.

**Lemma 2.1.** *If* $\mathrm{Supp}(\mu)$ *is not contained in a compact subgroup of* $G$, *then with probability* 1 *the sequence* $(|X_n(v_0)|)$ *is unbounded.*

*Proof.* As usual, we may suppose that $(\Xi, \mathscr{F}, P) = (G^{\mathbf{N}}, \mathscr{B}_G^{\mathbf{N}}, \mu^{\mathbf{N}})$ and the $Y_j$'s are the coordinate functions (use, e.g., [2, Prop. 2.39]). The left shift $t$ on $\Xi$ is measure-preserving and ergodic. Now $\Xi' = \{\xi \in \Xi\colon (|X_n(v_0)|) \text{ is bounded}\}$ is a $t$-invariant set, and so $P(\Xi') = 0$ or 1. Suppose that $P(\Xi') = 1$. Let $U$ be a compact neighbourhood of 1 in $G$. Let $S$ be a countable dense subset of $\mathrm{Supp}(\mu)$. Then for each $m \geq 1$ and each $(g_1, \dots, g_m) \in S^m$, let $V$ be a neighbourhood of 1 such that $g_1 V g_2 V \cdots g_m V \subset g_1 g_2 \cdots g_m U$. Let

$$A = \{\xi = (y_1, y_2, \dots) \in \Xi\colon y_j \in g_j V \text{ for } j = 1, \dots, m\}.$$

Then $P(A) = \prod_{j=1}^m \mu(g_j V) > 0$. By the Ergodic Theorem,

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_A(t^k \xi) \to P(A) \tag{2.1}$$

with probability 1, say for $\xi \in \Xi_{m, g_1, \dots, g_m}$. Let

$$\Xi'' = \bigcap_{m \geq 1, g_1, \dots, g_m \in S} \Xi_{m, g_1, \dots, g_m}.$$

Then $P(\Xi'') = 1$. Now let $\xi \in \Xi' \cap \Xi''$. Since $\xi \in \Xi'$, there is a compact set $K = K_\xi \subset G$ such that $X_n(\xi) \in K$ for all $n$. If $m \geq 1$ and $g_1, \dots, g_m \in S$, then (2.1) implies that $t^k\xi \in A$ infinitely often, and so $Y_{k+1}(\xi) \cdots Y_{k+m}(\xi) \in g_1 \cdots g_m U$ for infinitely many $k$. Hence $Y_{k+1}(\xi) \cdots Y_{k+m}(\xi) \in K^{-1}K \cap g_1 \cdots g_m U$, so that $g_1 \cdots g_m \in K^{-1}KU^{-1}$. This implies that the closed semigroup generated by $\mathrm{Supp}(\mu)$ is compact, and hence is a group [3]. This contradiction to our hypothesis shows that $P(\Xi') = 0$.

**Lemma 2.2.** *Let* $(\varphi_n)$ *be a sequence in* $G$ *such that* $\varphi_n(v_0) \to \omega_o \in \Omega$ *and* $\varphi_n^{-1}(v_0) \to \omega^* \in \Omega$. *Then* $\varphi_n\omega \to \omega_0$ *for each* $\omega \in (T \cup \Omega) \setminus \{\omega^*\}$.

*Proof*. Let us first observe that if $\varphi \in G$, $v \in T$ and $j(v, \varphi^{-1}(v_0)) = k < m = |\varphi(v_0)| = |\varphi^{-1}(v_0)|$, then $j(\varphi(v), \varphi(v_0)) = m - k$. Because if $(v_0, v_1, \cdots, v_m = \varphi^{-1}(v_0))$ and $(w_0 = v_0, w_1, \ldots, w_r = v)$ are the geodesics from $v_0$ to $\varphi^{-1}(v_0)$ and $v$ respectively, then $v_j = w_j$ for $j \le k$, while $v_{k+1} \ne w_{k+1}$. Then

$$(v_0 = \varphi(v_m), \varphi(v_{m-1}), \ldots, \varphi(v_k), \varphi(w_{k+1}), \ldots, \varphi(w_r) = \varphi(v))$$

is the geodesic from $v_0$ to $\varphi(v)$, while $(v_0 = \varphi(v_m), \ldots, \varphi(v_j), \ldots, \varphi(v_0))$ is that from $v_0$ to $\varphi(v_0)$.

Now let $\omega \in (T \cup \Omega) \setminus \{\omega^*\}$. Let $(v_0, v_1, \ldots)$ be the geodesic from $v_0$ to $\omega$. Then $j = j(\omega, \omega^*) < \infty$. Once $j(\varphi_n^{-1}(v_0), \omega^*) > j$, we have $j(\varphi_n^{-1}(v_0), v_k) \le j$ for all $k$. Hence, by the above, $j(\varphi_n(v_k), \varphi_n(v_0)) \ge |\varphi_n(v_0)| - j$. Thus given $N$, once $j(\varphi_n(v_0), \omega_0) \ge N$, $j(\varphi_n^{-1}(v_0), \omega^*) > j$ and $|\varphi_n(v_0)| \ge N + j$, we have $j(\varphi_n(v_k), \omega_0) \ge N$ for all $k$, so that $j(\varphi_n \omega, \omega_0) \ge N$.

**Theorem.** *Let $\mu$ be a regular Borel probability on the group $G$ of automorphisms of a locally finite infinite tree $T$. Let $(X_n)_{n \ge 0}$ be a corresponding right random walk on $G$. Assume that* Supp$(\mu)$ *is not contained in any amenable subgroup of $G$. Then with probability* 1 *there is an end $\omega \in \Omega$ such that $X_n(v) \to \omega$ in $T \cup \Omega$ for each fixed $v \in T$.*

*Proof*. Consider the induced action of $G$ on $\Omega$. Since $\Omega$ is compact, there is a probability $\nu$ on $\Omega$ such that $\mu * \nu = \nu$, i.e.,

$$\int_\Omega f(\omega)\, d\nu(\omega) = \int_\Omega \int_G f(g\omega)\, d\mu(g)\, d\nu(\omega)$$

for all continuous $f$ on $\Omega$. The nonamenability hypothesis implies that $\nu$ is a *continuous* measure. For if $\nu(\{\omega\}) > 0$ for some $\omega \in \Omega$, let $a = \max\{\nu(\{\omega\}) \colon \omega \in \Omega\}$ and $S = \{\omega \in \Omega \colon \nu(\{\omega\}) = a\}$. Then $S$ is finite, and $gS = S$ for all $g \in \mathrm{Supp}(\mu)$ because $\mu * \nu = \nu$. But Nebbia [12] showed that if $\omega_0 \in \Omega$, then the closed group $G_{\omega_0} = \{g \in G \colon g\omega_0 = \omega_o\}$ is amenable. Thus if $\omega_o \in S$, then $G_S = \{g \in G \colon g\omega = \omega \text{ for all } \omega \in S\}$ is a closed subgroup of $G_{\omega_0}$, and hence amenable. But $H = \{g \in G \colon gS = S\}$ contains $G_S$ as a normal subgroup of finite index, and is therefore amenable too. So $\mathrm{Supp}(\mu) \subset H$ is contrary to our hypothesis.

According to [10, Lemma 3.1(Corollary)], with probability 1, $X_n \nu$ converges weak*. That is, there is a set $N \subset \Xi$ with $P(N) = 0$, so that if $\xi \in \Xi \setminus N$, there is a probability $\lambda_\xi$ on $\Omega$ so that

$$\int_\Omega f(X_n(\xi)\omega)\, d\nu(\omega) \to \int_\Omega f(\omega)\, d\lambda_\xi(\omega)$$

for each $f$ in the space $\mathscr{C}(\Omega)$ of continuous functions on $\Omega$. We shall show that with probability 1, $\lambda_\xi$ is a Dirac measure $\delta_{\omega_0(\xi)}$.

By Lemma 2.1, there is a null set $N_1$ so that $(|X_n(\xi)v_0|)$ is unbounded if $\xi \notin N_1$. Fix $\xi \notin N \cup N_1$. Let $\omega_0 \in \Omega$ be a cluster point of $(X_n(\xi)v_0)$. Then we may choose a subsequence $(\varphi_j)$ of $(X_n(\xi))$ such that $\varphi_j(v_0) \to \omega_0$ and $\varphi_j^{-1}(v_0) \to \omega^*$ for some $\omega^* \in \Omega$. Then $\varphi_j\omega \to \omega_0$ for all $\omega \in \Omega \setminus \{\omega^*\}$ by Lemma 2.2. But

$$\int_\Omega f(\varphi_j\omega)\,d\nu(\omega) \to \int_\Omega f(\omega)\,d\lambda_\xi(\omega)$$

for all $f \in \mathscr{C}(\Omega)$, and $\nu\{\omega^*\} = 0$. Hence $\lambda_\xi = \delta_{\omega_0}$. This shows that $\omega_0$ is the only cluster point of $(X_n(\xi)v_0)$ in $\Omega$. Once we have checked that $|X_n(\xi)v_0| \to \infty$, it will then follow that $X_n(\xi)v_0 \to \omega_0$, and hence that $X_n(\xi)v \to \omega_0$ for all $v \in T$. But if $|X_n(\xi)v_0| \nrightarrow \infty$, there is a subsequence $(\varphi_j)$ of $(X_n(\xi))$ and there is a $\varphi \in G$ such that $\varphi_j \to \varphi$ in $G$. Then $\varphi_j\omega \to \varphi\omega$ for all $\omega \in \Omega$. But this implies that $\nu = \delta_{\varphi^{-1}\omega_0}$, which contradicts the continuity of $\nu$.

**Corollary 2.3.** *Let $\mu$ be a probability on the free group $\Gamma$ satisfying* (1.1). *Let $(X_n)$ be the corresponding right random walk on $\Gamma$. Then with probability* 1 *there is an end $\omega \in \Omega$ such that $X_n \to \omega$ in $\Gamma \cup \Omega$.*

*Remarks.* 1. Corollary 2.3 implies that $\Omega$ is a boundary of $\Gamma$ in the sense of [10]. Furthermore, if $T$ and $\mu$ are as in the Theorem, then $\Omega$ is a boundary of $G$ in the same sense. For we can put a topology on $G \cup \Omega$ making it compact with $G$ as an open dense subset. Indeed, the sets

$$N_k = \{\varphi \in G\colon j(\varphi(v_0), \omega) \geq k\} \cup \{\omega' \in \Omega\colon j(\omega', \omega) \geq k\},$$

for $k = 1, 2, \ldots,$ form a base of neighbourhoods at $\omega \in \Omega$ for such a topology.

2. Let $\mu$ be as in the Theorem or Corollary 2.3. Then there is only one probability $\nu$ on $\Omega$ such that $\mu * \nu = \nu$. For if $Z = \lim_{n\to\infty} X_n \in \Omega$, then $\nu$ must be the distribution of $Z$ (see [10, p. 18]).

## 4. Application to random walks in $\mathrm{PGL}(2, F)$, $F$ a local field

The general reference for this section is [14, Chapter II]. Let $F$ be a commutative field with a discrete valuation $v$. Let $\pi \in F$ satisfy $v(\pi) = 1$, and let $\mathscr{O} = \{x \in F\colon v(x) \geq 0\}$. Assume that $F$ is complete for the metric associated with $v$ and that the residue field $\mathscr{O}/\pi\mathscr{O}$ has $q < \infty$ elements, so that $F$ is locally compact. Let $G_0 = \mathrm{GL}(2, F)$, $Z = \{\lambda I\colon \lambda \in F^\times\}$ and $K = \{g \in G_0\colon g$ and $g^{-1}$ have entries in $\mathscr{O}\}$. Then $\widetilde{G}_0 = \mathrm{PGL}(2, F) = G_0/Z$ is a subgroup of the group $G$ of automorphisms of a homogeneous tree $T$ of degree $q + 1$. Indeed, $T$ may be realized as the set of equivalence classes of lattices in the two-dimensional vector space $F^2$, where two lattices $L$, $L'$ are equivalent if $L' = \lambda L$ for some $\lambda \in F^\times$. Let $\{e_1, e_2\}$ be the usual basis of $F^2$, let $L_0 = \mathscr{O}e_1 \oplus \mathscr{O}e_2$, and let $v_0$ be the class of $L_0$. Then $G_0$ acts transitively on $T$, and $KZ$ is the stabilizer of $v_0$. Thus $T \cong G_0/KZ$. The space $\Omega$ of ends of $T$ is isomorphic as a $G_0$-space to the set $\mathbf{P}_1(F)$ of lines in $F^2$ [14, p. 72]. The stabilizer of $l_0 = Fe_1 \in \mathbf{P}_1(F)$ is $B = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0 : c = 0\}$. So we may identify $\Omega$ with $G_0/B$. If $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Z \in \widetilde{G}_0$, then $\binom{\overline{a}}{c}$ and $\binom{\overline{b}}{d} \in \mathbf{P}_1(F)$ (where $\binom{\overline{a}}{c} = F(ae_1 + ce_2)$) are called the *columns* of $x$. If $\max(|a|, |c|) \geq \max(|b|, |d|)$, where $|t| = q^{-v(t)}$, i.e., if $\min(v(a), v(c)) \leq \min(v(b), v(d))$, we call $\binom{\overline{a}}{c}$ the *larger* of the columns of $x$, otherwise we call $\binom{\overline{b}}{d}$ the larger.

**Proposition 3.1.** *Let $\mu$ be a regular Borel probability on $\widetilde{G}_0$. Suppose that* $\operatorname{Supp}(\mu)$ *is not contained in an amenable subgroup of $\widetilde{G}_0$. Let $(X_k)_{k\geq 0}$ be the associated right random walk on $\widetilde{G}_0$. Then with probability* 1, *the larger of the two columns of $X_k$ tends to a limit in $\mathbf{P}_1(F)$.*

*Proof.* This will be an immediate consequence of the Theorem once we interpret what it means that $g_k KZ \to gB$ in $T \cup \Omega$. Now $l_0$ corresponds to the end $\omega_0$ of the chain $(v_0, v_1, v_2, \dots)$ in $T$, where $v_n$ is the class of the lattice $L_n = \mathscr{O}e_1 + \pi^n L_0 = \mathscr{O}e_1 \oplus \pi^n \mathscr{O}e_2 = x_n L_0$ for

$$x_n = \begin{pmatrix} 1 & 0 \\ 0 & \pi^n \end{pmatrix}.$$

For each $g \in G_0$ we have $gKZ = g'KZ$, where

$$g' = \begin{pmatrix} 1 & 0 \\ \tau & \pi^\delta \end{pmatrix} \quad \text{or} \quad g' = \begin{pmatrix} \tau & \pi^\delta \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

Here $\delta \geq 0$ is unique and $\tau \in \mathscr{O}$ is unique $\bmod \pi^\delta$ within each of the two types in (3.1), and

$$\begin{pmatrix} 1 & 0 \\ \tau & \pi^\delta \end{pmatrix} KZ = \begin{pmatrix} \tau' & \pi^\delta \\ 1 & 0 \end{pmatrix} KZ$$

can only happen for $\tau = \tau' = \delta = 0$ or when $\delta > 0$, $v(\tau) = v(\tau') = 0$ and $\tau' = \tau^{-1} \bmod \pi^\delta$. Also, $\delta = d(v_0, gv_0)$. Using this to calculate $d(gv_0, v_n) = d(v_0, g^{-1}x_n v_0)$, we find that $j(gv_0, \omega)$ (measured from $v_0$) equals $j(gv_0, v_n)$ once $n \geq \delta$, and this equals either $v(\tau)$, $\delta$, or 0, according as $g'$ in (3.1) is of the first type with $\tau \notin \pi^\delta \mathscr{O}$, the first type with $\tau \in \pi^\delta \mathscr{O}$, or the second type, respectively.

Thus $g_k KZ \to gB$ in $T \cup \Omega$ means that for large $k$,

$$g^{-1} g_k KZ = \begin{pmatrix} 1 & 0 \\ \tau_k & \pi^{\delta_k} \end{pmatrix} KZ,$$

where $m_k = \min(v(\tau_k), \delta_k) \to \infty$. So if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can find $\lambda_k \in F^\times$ and

$$\begin{pmatrix} t_k & u_k \\ t'_k & u'_k \end{pmatrix} \in K$$

so that

$$\begin{aligned}\lambda_k^{-1} g_k &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau_k & \pi^{\delta_k} \end{pmatrix} \begin{pmatrix} t_k & u_k \\ t'_k & u'_k \end{pmatrix} \\ &= \begin{pmatrix} at_k & au_k \\ ct_k & cu_k \end{pmatrix} \quad \bmod \pi^{m_k+\alpha},\end{aligned}$$

where $\alpha = \min(v(b), v(d))$. Since $\min(v(t_k),\ v(u_k)) = 0$, this proves the result.

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