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COMPLETE MÖBIUS STRIPS MINIMALLY IMMERSED IN $\mathbf{R}^3$

TORU ISHIHARA

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ABSTRACT. In the present paper, we will determine all complete minimal immersions of a Möbius strip into $\mathbf{R}^3$ with finite total curvature.

1. INTRODUCTION

Let M be a Möbius strip. In the sequel, we will study a regular complete minimal immersion $x: M \to \mathbf{R}^3$ with finite total curvature. Examples of such immersions with total curvature $-2\pi m$, for any odd integer $m \geq 3$ was obtained by Meeks [3] (case $m = 3$) and Oliveira [2] (case $m \geq 3$). Barros [1] has investigated minimal immersions with total curvature -10π. Since M may be identified with the quotient space of $\widetilde{M} = \mathbf{C} - \{0\}$ by the equivalence relation $z \sim I(z)$, where I is the transformation given by $I(z) = -1/\overline{z}$, the natural projection $\pi: \widetilde{M} \to M$ is the oriented two-sheeted covering of M. It is shown by Meeks [3] that for an immersion $\tilde{x}: \widetilde{M} \to \mathbf{R}^3$, there exists a minimal immersion $x: M \to \mathbf{R}^3$ such that $\tilde{x} = x \bullet \pi$ if and only if $\tilde{x}(I(z)) = \tilde{x}(z)$ for each $z \in \widetilde{M}$. In this case, the immersion $\tilde{x}: \widetilde{M} \to \mathbf{R}^3$ is called a double surface of $x: M \to \mathbf{R}^3$.

Let f and g be the functions of Weierstrass representation of a complete regular minimal immersion $\tilde{x}: \widetilde{M} \to \mathbf{R}^3$, that is,

$$(1) \qquad \tilde{x}(p) = \frac{1}{2}\operatorname{Re}\int_{p^0}^{p} f(z)(1 - g^2(z), i(1 + g^2(z)), 2g(z))\,dz, \quad p, p_0 \in \widetilde{M}.$$

It is shown by Meeks [3] (see also [2]) that the immersion $\tilde{x}$ is a double surface if and only if the functions f and g satisfy

$$(2) \qquad g(I(z)) = -1/\overline{g(z)}, \quad \overline{f(I(z))} = -f(z)(zg(z))^2 \qquad (z \in \widetilde{M}).$$

The function g is regarded as the Gauss map of the immersion. Oliveira shows [2] that the total curvature of a complete regular immersion of M in $\mathbf{R}^3$

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with finite total curvature is of the form $C(M) = -2\pi m$, m odd, $m \geq 3$ and that in this case, the Gauss map g of the immersion is of the form

$$(3)\quad g(z) = bz^{m-n}\prod_{j=1}^{n}(z-a_j)/\prod_{j=1}^{n}(\overline{a}_j z+1) \qquad \left(m \geq n,\ \prod_{j=1}^{n} a_j \neq 0, |b| = 1\right).$$

After a rotation, we can put $b = 1$. From (2) and (3), we can show easily that in this case, the other function f of the Weierstrass representation has the form

$$(4)\qquad f(z) = i\lambda\overline{b}\prod_{j=1}^{n}(\overline{a}_j z+1)^2/z^{m+1} \qquad (\lambda > 0).$$

2. Main results

The integrals in the formula (1) must have no real periods, that is, the real parts of the integrals along any closed path must be zero. The following lemma is essential in the present work and shall be proved later.

Lemma. *Let $a_1, \ldots, a_n$ and b be complex numbers such that $\prod_{j=1}^{n} a_j \neq 0$ and $|b| = 1$. Let λ be a positive real number. Let $S_1, \ldots, S_n$ be the elementary symmetric polynomials of $a_1, \ldots, a_n$. Put $S_0 = 1$. Assume that the functions g and f are given by* (3) *and* (4) *respectively. Then, the integrals in the formula* (1) *have no real periods if and only if $S_0, S_1, \ldots, S_n$ satisfy the following conditions* (5) *and* (6):

$$(5)\qquad \sum_{i=0}^{n}(-1)^i|S_i|^2 = 0,$$

$$(6)\qquad \sum_{i+j=m} S_i S_j = 0.$$

The equality (5) does not hold for $n = 0$. Hence the n is always greater than 0. When $m > 2n > 0$, the condition (6) has no meaning. From this lemma, we obtain our main result.

Theorem. *Let M be a Möbius strip and $\pi\colon \widetilde{M} \to M$ be the two-sheeted covering, where $\widetilde{M} = \mathbf{C} - \{0\}$. Let f and g be the functions on $\widetilde{M}$ given by* (3) *and* (4) *respectively, where m is odd, $m \geq 3$, $m \geq n \geq 1$ and $a_1, \ldots, a_n$, b, λ satisfy the conditions in the above lemma. Then the Weierstrass representation formula* (1) *gives a minimal immersion $x\colon \widetilde{M} \to \mathbf{R}^3$ from which we obtain a regular complete minimal immersion $x\colon M \to \mathbf{R}^3$ such that $\tilde{x} = x \bullet \pi$. The total curvature of the immersion x is $-2\pi m$. On the other hand, any regular complete minimal immersion of M into $\mathbf{R}^3$ is given in this way.*

3. Proof of lemma

Put

$$F_1 = f(1-g^2)/2, \qquad F_2 = if(1+g^2)/2, \qquad F_3 = fg,$$

and

$$P = \prod_{j=1}^{n}(\overline{a}_j z + 1), \qquad Q = z^{m-n}\prod_{j=1}^{n}(z - a_j), \qquad R = z^{m+1}.$$

Then we have

$$F_1 = i\lambda(\overline{b}P^2 - bQ^2)/(2R), \qquad F_2 = -\lambda(\overline{b}P^2 + bQ^2)/(2R), \qquad F_3 = i\lambda PQ/R.$$

For each i $(1 \le i \le 3)$, the integral $\int F_i\,dz$ has no real periods if and only if the imaginary part of the residue of F_i at the origin 0 vanishes, that is,

$$\operatorname{Im}\operatorname{Res}_0 F_i = 0 \qquad (1 \le i \le 3). \tag{7}$$

Set $F_i = G_i/R$ $(1 \le i \le 3)$, and $G_i = \sum_{j=0}^{2m} A_{(i)j} z^j$. Then we have

$$\operatorname{Res}_0 F_i = A_{(i)m}. \tag{8}$$

When $2n < m$, as the degree of $P^2 \le 2n < m$ and the order of $Q^2 \ge m+1$, it follows that

$$\operatorname{Res}_0 F_1 = \operatorname{Res}_0 F_2 = 0 \qquad (2n < m). \tag{9}$$

Next, assume $2n \ge m$. Then we have

$$Q^2 = z^{2(m-n)}\sum_{k=0}^{2n}\sum_{i+j=k}(-1)^k S_i S_j z^{2n-k}.$$

Hence we get that the coefficient of z^m is $-\sum_{i+j=m} S_i S_j$. Similarly, concerning the polynomial P^2, the coefficient of z^m is $\sum_{i+j=m}\overline{S}_i\overline{S}_j$. By setting $A = b\sum_{i+j=m} S_i S_j$, we have

$$A_{(1)m} = i\lambda(A + \overline{A})/2 = i\lambda\operatorname{Re}(A), \qquad A_{(2)m} = \lambda(A - \overline{A})/2 = i\lambda\operatorname{Im}(A). \tag{10}$$

Thus we obtain that $\operatorname{Im}\operatorname{Res}_0 F_1 = 0$ and $\operatorname{Im}\operatorname{Res}_0 F_2 = 0$ if and only if the condition (6) holds.

At last, we will calculate $\operatorname{Res}_0 F_3$. From the equality

$$PQ = z^{m-n}\sum_{k=0}^{2n}\sum_{n-i+j=k}(-1)^i S_i\overline{S}_j z^k,$$

we obtain

$$A_{(3)m} = i\lambda\sum_{i=0}^{n}(-1)^i|S_i|^2. \tag{11}$$

Hence $\operatorname{Im}\operatorname{Res}_0 F_3 = 0$ if and only if the condition (5) holds.

4. Further results

Similarly as in [1], we denote by $\operatorname{ord}(g)$, the order of the ramification of the Gauss map g at the end of M. We can put

$$\operatorname{ord}(g) = m - n \qquad (m \ge n \ge 1).$$

When $\operatorname{ord}(g) = m - 1$, the condition in the lemma reduces to $1 - |S_1|^2 = 1 - |a_1|^2 = 0$. Hence, after a rotation of the coordinate of $\mathbf{C} - \{0\}$, we can put $a_1 = 1$.

Corollary. *Let* $x: M \to \mathbf{R}^3$ *be a complete minimal immersion of the Möbius strip with total curvature* $-2\pi m$ $(m$ *odd,* $m \geq 3)$, *and* $\operatorname{ord}(g) = m - 1$. *Then it is essentially the minimal immersion given by Meeks* [3] $(m = 3)$ *and Oliveira* [2] $(m > 3)$.

In the case $m = 5$, the above is noticed at first by Barros [1]. Next, we apply our theorem to the case $m = 5$. We will obtain a_i's or S_i's which satisfy the conditions in the lemma.

Case 1. $\operatorname{ord}(g) = 3$. After a rotation of $\mathbf{C} - \{0\}$, we can put

$$a_1 = \lambda_1 e^{\sqrt{-1}\theta}, a_2 = \lambda_2 e^{-\sqrt{-1}\theta} \qquad (\lambda_1, \lambda_2 > 0,\ \pi \geq \theta \geq 0).$$

Substituting these into (5), we get $\cos 2\theta = -(\lambda_1 - 1/\lambda_1)(\lambda_2 - 1/\lambda_2)/2$. Hence it should be $|(\lambda_1 - 1/\lambda_1)(\lambda_2 - 1/\lambda_2)| \leq 2$. If $\lambda_1 = 1$, then $\theta = \pi/4$ or $3\pi/4$ and λ_2 is an arbitrary positive real number. This type is discussed by Barros in the proof of [1, Theorem 1.1].

Case 2. $\operatorname{ord}(g) = 2$. The condition (6) is reduced to $S_2 S_3 = 0$. As S_3 is not zero, $S_2 = 0$. Using (5), we can put

$$S_1 = \sqrt{1 - \mu^2} e^{\sqrt{-1}\omega}, \quad S_2 = 0, \quad S_3 = \mu \qquad (1 \geq \mu > 0,\ 2\pi > \omega \geq 0).$$

Case 3. $\operatorname{ord}(g) = 1$. Set $S_i = \mu_i e^{\sqrt{-1}\omega_i} (\mu_i > 0,\ 2\pi > \omega_i \geq 0)$, $(1 \leq i \leq 4)$. The condition (6) is now $S_2 S_3 + S_1 S_4 = 0$. Hence we have $u_2 u_3 = u_1 u_4$, $e^{\sqrt{-1}(\omega_2 + \omega_3)} + e^{\sqrt{-1}(\omega_1 + \omega_4)} = 0$. After a rotation, we may put $\omega_2 + \omega_3 = 0$. Then we have $\omega_1 + \omega_4 = \pi$ or 3π. Using (5), we obtain

$$\begin{aligned} S_1 &= \mu_1 e^{\sqrt{-1}\omega_1}, \qquad S_2 = \mu_2 e^{\sqrt{-1}\omega_2}, \quad S_3 = -\mu\mu_1 e^{-\sqrt{-1}\omega_2}, \\ S_4 &= \mu\mu_2 e^{-\sqrt{-1}\omega_1}, \quad \mu = \sqrt{1/(\mu_1^2 - \mu_2^2) - 1} \\ &(\mu_1, \mu_2 > 0,\ 1 > \mu_1^2 - \mu_2^2 > 0,\ 2\pi > \omega_1, \omega_2 \geq 0). \end{aligned}$$

Case 4. $\operatorname{ord}(g) = 0$. Set $S_i = \mu_i e^{\sqrt{-1}\omega_i} (\mu_i > 0, 2\pi > \omega_i \geq 0)$, $(1 \leq i \leq 5)$. Then we have $1 + \mu_2^2 + \mu_4^2 = \mu_1^2 + \mu_3^2 + \mu_5^2$. We can assume

$$\omega_5 = \omega_2 + \omega_3 \quad \text{and} \quad \omega_4 = \omega_2 + \omega_3 - \omega_1 + \omega, \qquad \text{where } \omega = \pm\pi \text{ or } \pm 3\pi.$$

Hence we get $\mu_5 = \mu_1\mu_4 - \mu_2\mu_3$.

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Department of Mathematics and Computer Sciences, Tokushima University, Tokushima 770, Japan