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THE FIXED-POINT-SPACE DIMENSION FUNCTION FOR A FINITE GROUP REPRESENTATION

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ABSTRACT. Given a complex representation of a finite group G, construct the integer valued function α on G by setting $\alpha(g)$ to be the dimension of the fixed-point-space of g in the module corresponding to the given representation. Usually, α is not a generalized character of G and for trivial reasons $|G|\alpha$ is always a generalized character. The main result of this paper is that $e\alpha$ is always a generalized character, where e is the exponent of G.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let G be a finite group and let α be a $\mathbf{Z}$-valued function on G which is constant on rational-classes, (i.e. $\alpha(x) = \alpha(y)$ if the cyclic groups $\langle x\rangle$ and $\langle y\rangle$ are G-conjugate.) Although α need not be a generalized character (i.e. a $\mathbf{Z}$-linear combination of $\mathrm{Irr}(G)$), it is always true that $|G|\alpha$ is a generalized character. (This follows immediately from the observation that for each $\chi \in \mathrm{Irr}(G)$, the sum of the values over each rational-class lies in $\mathbf{Z}$.)

For certain natural functions, α, a multiplier m smaller than $|G|$ is sufficient to make $m\alpha$ a generalized character. Our main result is that if $\alpha(x)$ denotes the dimension of the fixed-point space of x in some $\mathbf{C}$-representation of G, then $e\alpha$ is a generalized character, where e is the exponent of G.

If $\mathscr{X}$ is a $\mathbf{C}$-representation of G affording the character χ, then the fixed-point-space dimension function α associated with $\mathscr{X}$ can be computed from χ by the formula $\alpha(x) = [\chi_{\langle x\rangle}, 1_{\langle x\rangle}]$, where $[\cdot,\cdot]$ denotes the character inner product. We introduce the notation $\hat{\chi}$ for this function.

Theorem 1. *Let χ be any character of G. Then $e\hat{\chi}$ is a generalized character, where e is the exponent of G and $\hat{\chi}(x) = [\chi_{\langle x\rangle}, 1_{\langle x\rangle}]$ for $x \in G$.*

In order to prove that $e\hat{\chi}$ is a generalized character of G, it suffices by Brauer's charaterization of characters to show that the restriction $(e\hat{\chi})_N$ is a generalized character of N for every nilpotent subgroup $N \subseteq G$. Because

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$(\hat{\chi})_N = \hat{\chi}_N$ and the exponent of N divides that of G, it is enough to prove Theorem 1 in the case that G is nilpotent.

We need to show (when G is nilpotent) that $e[\hat{\chi}, \zeta] \in \mathbf{Z}$ for all irreducible characters ζ of N. In order to make our inductive proof of this work, we need to generalize it.

Proposition 2. *Let G be nilpotent and suppose χ is a character of G and ζ is a generalized character. Then $m[\hat{\chi}, \zeta] \in \mathbf{Z}$, where m is any positive integer such that $\zeta(x) = 0$ whenever $x^m \neq 1$.*

In particular, the condition that $\zeta(x) = 0$ whenever $x^m \neq 1$ is vacuously satisfied if $m = e$, the exponent of G. Proposition 2, therefore, includes the nilpotent group case of Theorem 1 and so implies the general case by Brauer's theorem.

2. Proofs

If χ is a complex-valued function on an arbitrary finite group G and n is a positive integer, we write $\chi^{(n)}$ to denote the function on G defined by $\chi^{(n)}(x) = \chi(x^n)$. It is well known that if χ is a character (or a generalized character) of G, then $\chi^{(n)}$ is a generalized character. (See, for instance, [1, Problem 4.7].)

Somewhat analogously, we define $\chi^{(1/n)}$ by the formula

$$\chi^{(1/n)}(x) = \sum_{\substack{y \in G \\ y^n = x}} \chi(y).$$

Lemma 3. *If χ is a generalized character of G and n is a positive integer, then $\chi^{(1/n)}$ is a generalized character.*

Proof. Let $\zeta \in \mathrm{Irr}(G)$. Then

$$\begin{aligned}
[\chi^{(1/n)}, \zeta] &= \frac{1}{|G|} \sum_{x \in G} \chi^{(1/n)}(x)\overline{\zeta(x)} \\
&= \frac{1}{|G|} \sum_{x \in G} \sum_{\substack{y \in G \\ y^n = x}} \chi(y)\overline{\zeta(x)} \\
&= \frac{1}{|G|} \sum_{y \in G} \chi(y)\overline{\zeta(y^n)} \\
&= [\chi, \zeta^{(n)}].
\end{aligned}$$

Since $\zeta^{(n)}$ is a generalized character, we have $[\chi, \zeta^{(n)}] \in \mathbf{Z}$ and it follows that $\chi^{(1/n)}$ is a generalized character. ■

We need one further lemma for our proof of Proposition 2.

Lemma 4. *Let $N \triangleleft G$ with G/N cyclic, and suppose χ is a generalized character of G such that $\chi(x) = 0$ for all $x \in G - N$. Then $\chi = \psi^G$ for some generalized character ψ of N.*

Proof. Let $C = \mathrm{Irr}(G/N)$ so that C may be viewed as a group of linear characters of G and C acts on $\mathrm{Irr}(G)$ by multiplication. If $\lambda \in C$, then $\chi = \chi\lambda$ since $\chi(x) = 0$ if $\lambda(x) \neq 1$. It follows that χ is a **Z**-linear combination of sums of orbits of the action of C on $\mathrm{Irr}(G)$ and so it suffices to show that each orbit sum is induced from some character of N.

Let $\xi \in \mathrm{Irr}(G)$ and let η be the sum of the C-orbit containing ξ. If α is any irreducible constituent of ξ_N, we will complete the proof by showing that $\alpha^G = \eta$.

If $x \in G - N$, we can choose $\lambda \in C$ with $\lambda(x) \neq 1$. Since η is a C-orbit sum, we have $\eta\lambda = \eta$ and so $\eta(x) = 0$. Also, $\alpha^G(x) = 0$ and so η and α^G agree on $G - N$.

Each irreducible constituent of η has the form $\xi\mu$ for some $\mu \in C$. Since $\mu_N = 1_N$, we have $(\xi\mu)_N = \xi_N$ and thus η_N is a multiple of ξ_N which is a multiple of the sum of the G-orbit of α. Also, $(\alpha^G)_N$ is a multiple of the G-orbit sum of α and thus, to show that η and α^G agree on N (and thereby complete the proof), it suffices to compare degrees and show that $\eta(1) = \alpha^G(1)$.

Let $B \subseteq C$ be the stabilizer of ξ. Then $\eta(1) = |C\colon B|\xi(1)$. If $T \subseteq G$ is the stabilizer of α, then $\xi = \beta^G$ where $\beta \in \mathrm{Irr}(T)$ and β_N is a multiple of α. Since T/N is cyclic, we have $\beta_N = \alpha$ and so $\xi(1) = |G\colon T|\alpha(1)$ and $\eta(1) = |C\colon B||G\colon T|\alpha(1)$. Of course, $\alpha^G(1) = |G\colon N|\alpha(1)$ and so we must show that $|G\colon N| = |G\colon T||C\colon B|$. It suffices, therefore, to prove that $|T\colon N| = |C\colon B|$.

We claim that $B = \mathrm{Irr}(G/T)$. This would give $|C\colon B| = |C|/|B| = |G\colon N|/|G\colon T| = |T\colon N|$, as required. To prove the claim, let $\mu \in B$. Then

$$\xi = \xi\mu = \beta^G\mu = (\beta\mu_T)^G$$

and $(\beta\mu_T)_N = \beta_N = \alpha$. However, β is the unique character of T lying over α which induces ξ, and thus $\beta\mu_T = \beta$. Since β_N is irreducible and $N \subseteq \ker(\mu_T)$, it follows that $\mu_T = 1_T$. (Use [1, 6.17], for instance). Thus $\mu \in \mathrm{Irr}(G/T)$ as required.

Conversely, suppose $\mu \in \mathrm{Irr}(G/T)$. Then

$$\xi\mu = \beta^G\mu = (\beta\mu_T)^G = \beta^G = \xi$$

and so $\mu \in B$. The proof is now complete. ■

Proof of Proposition 2. The map $\chi \mapsto \hat{\chi}$ is additive and so it is no loss to assume that χ is an irreducible character. We proceed by induction on $\chi(1)$.

First, suppose $\chi(1) = 1$ so that χ is linear. Writing $K = \ker(\chi)$, we see that $\hat{\chi}(x) = 0$ if $x \notin K$ and $\hat{\chi}(x) = 1$ if $x \in K$. Let $N = \{g \in G | g^m \in K\}$, the preimage in G of the set of elements in G/K with order dividing m. Since G/K is cyclic, it follows that N is a subgroup and $|N/K|$ divides m. Thus

$m\hat{\chi}_N$ is a multiple of $|N/K|\hat{\chi}_N = (1_K)^N$. In particular, $m\hat{\chi}_N$ is a character of N.

If $x \in G - N$, then $x^m \notin K$ and so $x^m \neq 1$ and $\zeta(x) = 0$ by hypothesis. Since G/N is cyclic, Lemma 4 tells us that $\zeta = \eta^G$ some generalized character η of N. Then $m[\hat{\chi}, \zeta] = [m\hat{\chi}, \eta^G] = [m\hat{\chi}_N, \eta]$ and this lies in $\mathbf{Z}$ since $m\hat{\chi}_N$ is a character.

Now assume $\chi(1) > 1$. The nilpotence of G guarantees that we can write $\chi = \psi^G$, where $\psi \in \mathrm{Irr}(H)$ and $H \lhd G$ has prime index p. Then $\chi_H = \psi_1 + \psi_2 + \cdots + \psi_p$ where $\psi_1 = \psi$ and the ψ_i constitute an orbit of the action of G on $\mathrm{Irr}(H)$. Thus $\hat{\chi}_H = \sum_i \hat{\psi}_i$ and the $\hat{\psi}_i$ are all G-conjugate. We will also need to be able to compute $\hat{\chi}(x)$ if $x \in G - H$; we work with modules to do this.

Let V be a $\mathbf{C}G$-module affording χ. Then $V = V_1 \dot{+} V_2 \dot{+} \cdots \dot{+} V_p$, where the V_i are $\mathbf{C}H$-submodules of V affording the characters $\psi_i \in \mathrm{Irr}(H)$ and these submodules are permuted by G. In fact, if $x \in G - H$, then $\langle x \rangle$ permutes the V_i transitively and we may assume x carries V_i to V_{i+1} for $1 \leq i < p$ and x carries V_p to V_1.

Holding $x \in G - H$ fixed, we can define a map $w \mapsto \tilde{w}$ from V_1 into V by setting $\tilde{w} = w + wx + wx^2 + \cdots + wx^{p-1}$. This is a $\mathbf{C}$-linear transformation which is injective since $\tilde{w} \neq 0$ when $w \neq 0$ because $wx^i \in V_{i+1}$ for $0 \leq i \leq p-1$ and the sum $\sum V_i$ is direct. If w is a fixed point of x^p in V_1, then clearly $\tilde{w}$ is fixed by x. Conversely, if $v \in V$ is a fixed point of x, then writing $v = v_1 + v_2 + \cdots + v_p$ with $v_i \in V_i$ and comparing the projections of the equal vectors v and vx into the V_i, we see that $v_1 x = v_2$, $v_2 x = v_3, \ldots, v_{p-1}x = v_p$ and $v_p x = v_1$. It follows that $v = \tilde{v}_1$ and that v_1 is fixed by x^p.

We have proved that the map $w \mapsto \tilde{w}$ defines an isomorphism of the fixed-point space of x^p in V_1 onto the fixed-point space of x in V. This shows that $\hat{\chi}(x) = \hat{\psi}(x^p)$ for all $x \in G - H$.

Now

$$m[\hat{\chi}, \zeta] = \frac{m}{|G|} \sum_{x \in G} \hat{\chi}(x)\overline{\zeta(x)} = \frac{m}{|G|} S_1 + \frac{m}{|G|} S_2$$

where S_1 is the sum for $x \in H$ and S_2 is the sum for $x \in G - H$. When $x \in H$, we have $\hat{\chi}(x) = \sum_i \hat{\psi}_i(x)$ and so

$$S_1 = |H| \left[\left(\sum_i \hat{\psi}_i \right), \zeta_H \right] = p|H|[\hat{\psi}, \zeta_H]$$

since all $[\hat{\psi}, \zeta_H]$ are equal. (This is because the $\hat{\psi}_i$ are G-conjugate). Thus $(m/|G|)S_1 = m[\hat{\psi}, \zeta_H]$ and this lies in $\mathbf{Z}$ by the inductive hypothesis since $\psi(1) < \chi(1)$.

To evaluate S_2, define $\xi(y) = \sum_{x \in G-H, x^p = y} \zeta(x)$. Then

$$\begin{aligned} S_2 &= \sum_{x \in G-H} \hat{\chi}(x)\overline{\zeta(x)} \\ &= \sum_{x \in G-H} \hat{\psi}(x^p)\overline{\zeta(x)} \\ &= \sum_{y \in H} \hat{\psi}(y)\overline{\xi(y)} \\ &= |H|[\hat{\psi}, \xi] \end{aligned}$$

and so $(m/|G|)S_2 = (m/p)[\hat{\psi}, \xi]$ and we need to show that this lies in $\mathbf{Z}$.

If $x \in G - H$ and $x^p = y \in H$, then p divides the order of x. If $p \nmid m$, then $x^m \neq 1$ and so $\zeta(x) = 0$ by hypothesis. It follows that ξ is identically zero if $p \nmid m$ and there is nothing to prove in this case. Suppose, therefore, that $p | m$. If $x^m = 1$ and $x^p = y$, then $y^{(m/p)} = 1$. Therefore, if $y^{(m/p)} \neq 1$, we have $\zeta(x) = 0$ for all x with $x^p = y$ and so $\xi(y) = 0$.

Now

$$\xi(y) = \sum_{\substack{x \in G \\ x^p = y}} \zeta(x) - \sum_{\substack{x \in H \\ x^p = y}} \zeta(x)$$

and so $\xi = (\zeta^{(1/p)})_H - (\zeta_H)^{(1/p)}$ and this is a generalzied character by Lemma 3. Since $\xi(y) = 0$ if $y^{(m/p)} \neq 1$, the inductive hypothesis yields $(m/p)[\hat{\psi}, \xi] \in \mathbf{Z}$, as required. ■

The proof of our main result, Theorem 1, is now complete.

3. Further remarks

If we drop the nilpotence hypothesis from Proposition 2, we get a statement properly stornger than Theorem 1. Is this statement true?

Theorem 5. *Let χ be a character of an arbitrary finite group G and let ζ be a generalized character of G with the property that $\zeta(x) = 0$ whenever $x^m \neq 1$ for some fixed positive integer m. Then $m[\hat{\chi}, \zeta] \in \mathbf{Z}$.*

Proof. By Brauer's theorem on induced characters, the principal character 1_G can be written as a sum of induced generalized characters α_i^G where the α_i are characters of certain nilpotent subgroups $N_i \subseteq G$. Then

$$\zeta = \zeta 1_G = \sum_i \zeta(\alpha_i)^G = \sum_i (\zeta_{N_i}\alpha_i)^G.$$

Thus

$$m[\hat{\chi}, \zeta] = \sum m[\hat{\chi}, (\zeta_{N_i}\alpha_i)^G] = \sum_i m[\hat{\chi}_{N_i}, \zeta_{N_i}\alpha_i]$$

and this is a sum of integers by Proposition 2, since the $\zeta_{N_i}\alpha_i$ are generalized characters which vanish on elements $x \in N_i$ such that $x^m \neq 1$. ■

We began this paper with the observation that if α is any $\mathbf{Z}$-valued function on G, constant on rational classes, then $|G|\alpha$ is a generalized character; equivalently $|G|[\alpha, \zeta] \in \mathbf{Z}$ for each $\zeta \in \mathrm{Irr}(G)$. The point of Theorem 1 is that in certain situations, for particular α, the coefficient $|G|$ can be reduced uniformly; i.e. for all ζ. It is true (and quite trivial) that for particular ζ we can reduce the coefficient for all α. We conclude with this observation.

Theorem 6. *Let α be $\mathbf{Z}$-valued and constant on rational-classes of G. Then*

$$\frac{|G|}{\zeta(1)}[\alpha, \zeta] \in \mathbf{Z}$$

for all $\zeta \in \mathrm{Irr}(G)$.

Proof. Let $x_1, x_2, \ldots, x_k$ be representatives for the conjugacy classes $K_1, \ldots, K_k$ of G. Then

$$\begin{aligned} \frac{|G|}{\zeta(1)}[\alpha, \zeta] &= \sum_{x \in G} \alpha(x) \frac{\overline{\zeta(x)}}{\zeta(1)} \\ &= \sum_{i=1}^{k} \alpha(x) \frac{\overline{\zeta(x_i)}}{\zeta(1)} |K_i| \end{aligned}$$

and this is an algebraic integer since $\zeta(x_i)|K_i|/\zeta(1)$ is integral. Also, $(|G|/\zeta(1))[\alpha, \zeta]$ is rational since $|G|[\alpha, \zeta] \in \mathbf{Z}$. The result follows. ■

In view of Theorems 1 and 6, one might guess that if $\chi, \zeta \in \mathrm{Irr}(G)$, then $(e/\zeta(1))[\hat{\chi}, \zeta] \in \mathbf{Z}$, where e is the exponent of G. This is false since if G has order p^3 and exponent p (where p is prime), then $[\hat{\chi}, \chi] = 1/p$ for $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = p$.

REFERENCES

1. I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.

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