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$O(2) \times O(2)$-INVARIANT HYPERSURFACES WITH CONSTANT NEGATIVE SCALAR CURVATURE IN E^4

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ABSTRACT. We use the method of equivariant differential geometry to prove the existence of a complete hypersurface with constant negative scalar curvature in $E^n (n \geq 4)$. This is the first example of a complete hypersurface with constant negative scalar curvature in $E^n (n \geq 4)$.

1. INTRODUCTION

Let (G, E^{n+1}) be an isometric transformation group of the $(n+1)$-dimensional Euclidean space E^{n+1} with codimension two principal orbit type. Such orthogonal transformation groups are classified in [2]. It is natural and interesting to study hypersurfaces in E^{n+1} with constant scalar curvature, which are invariant with respect to one of such orthogonal transformation group. In [1,4], $O(n)$-invariant hypersurfaces with constant scalar curvature in real space forms are classified completely. In particular, it is proved that all $O(n)$-invariant complete hypersurfaces with constant scalar curvature in E^{n+1} have nonnegative scalar curvature. In this paper, we consider $O(2) \times O(2)$-invariant hypersurfaces in E^4. We show that there is an $O(2) \times O(2)$-invariant complete hypersurface M^3 of constant negative scalar curvature in E^4. Making product M^3 with E^{n-3}, we obtain a complete hypersurface with constant negative scalar curvature in E^{n+1}. Note that these spaces are the first examples of complete hypersurfaces of constant negative scalar curvature in the Euclidean spaces.

2. $O(2) \times O(2)$-INVARIANT HYPERSURFACES

We consider $(O(2) \times O(2), E^4)$, where $O(2) \times O(2)$ acts orthogonally on E^4 via the representation $\rho_2 \oplus \rho_2'$. We need the following fact [3].

(i) The orbit space of $O(2) \times O(2)$-action on E^4 can be parametrized by $Q: = \{(x, y); x \geq 0, y \geq 0\}$ with the orbital distance metric $ds^2 = dx^2 + dy^2$.

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(ii) Interior points of Q correspond to principal orbits which are of the type $S^1 \times S^1$. More precisely, the inverse image of (x, y) in the interior of Q is a product $S^1(x) \times S^1(y)$.

We consider a smooth curve $\gamma(s) = (x(s), y(s))$ in the interior of Q, which is parametrized by the arc length (i.e. $(dx/ds)^2 + (dy/ds)^2 = 1$). We denote the $O(2) \times O(2)$-invariant hypersurface in E^4 generated by γ as M_γ. The induced metric g on M_γ is $g = x(s)^2 g_1 + y(s)^2 g_2 + ds^2$, where g_1, g_2 are the canonical metrics on $S^1(1)$, respectively.

Proposition 1. *M_γ is of constant scalar curvature a if and only if γ satisfies the following differential equation.*

$$(x'y'' - y'x'')\left(-\frac{x'}{y} + \frac{y'}{x}\right) - \frac{x'y'}{xy} = \frac{a}{2}, \tag{1}$$

where x', y', x'', y'' are derivatives with respect to s.

Proof. Since the principal curvatures of M_γ in E^4 are $x'y'' - y'x''$, $-x'/y$ and y'/x, we get the conclusion.

Using $(dx/ds)^2 + (dy/ds)^2 = 1$ and $dy/ds = (dy/dx)(dx/ds)$, we write (1) as

$$\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\left\{\frac{dy}{dx} + \frac{axy}{2}\left(1 + \left(\frac{dy}{dx}\right)^2\right)\right\}\left(y\frac{dy}{dx} - x\right)^{-1}. \tag{2}$$

From now on we assume that $a \leq 0$. We denote by $f(x, a)$ the solution of (2) satisfying the initial condition $y(1) = 1$, $y'(1) = -1$. Since (1) and (2) are symmetric with respect to x and y, we only consider $x \geq 1$ part of $y = f(x, a)$. From the elementary ODE theory, we see that $y = f(x, a)$ exists at least on the interval $[1, x']$, where x' is the least point which satisfies $f(x', a) = 0$ or $(\partial f/\partial x)(x', a) = 0$.

Lemma 1. *If $f(x, a) > 0$ and $(\partial f/\partial x)(x, a) < 0$, then $(\partial^2 f/\partial x^2)(x, a) > 0$.*

Proof. From $a \leq 0$ and (2) we easily get the conclusion.

Lemma 2. *If there is $x_0 > 1$ such that $f(x_0, a) = 0$ and $f(x, a) > 0$, $(\partial f/\partial x)(x, a) < 0$ for all $x \in [1, x_0)$, then $(\partial f/\partial x)(x_0, a) < 0$ (i.e. $y = f(x, a)$ intersects with the x-axis transversally).*

Proof. If $(\partial f/\partial x)(x_0, a) = 0$, then $y = 0$ and $y = f(x, a)$ are two distinct solutions of (2) with the same initial condition $y(x_0) = 0$, $(dy/dx)(x_0) = 0$. This is a contradiction.

By Lemmas 1 and 2, we can classify the solution $y = f(x, a)$ $(x \geq 1)$.

Lemma 3. *For any $a \leq 0$, $y = f(x, a)$ $(x \geq 1)$ is one of the following three types* (see Figure 1).

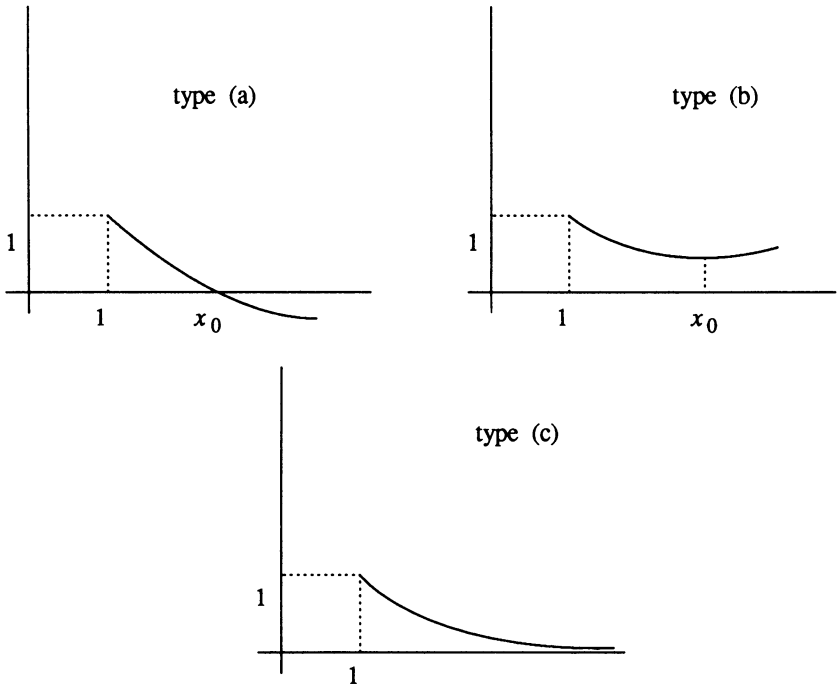

FIGURE 1.

(a) *There is* $x_0 > 1$ *such that* $f(x_0, a) = 0$ *and* $f(x, a) \geq 0$, $(\partial f/\partial x)(x, a) < 0$, $(\partial^2 f/\partial x^2)(x, a) > 0$ *for all* $x \in [1, x_0]$. *In this case* $y = f(x, a)$ *intersects with the* x*-axis transversally.*

(b) *There is* $x_0 > 1$ *such that* $(\partial f/\partial x)(x_0, a) = 0$ *and* $f(x, a) \geq f(x_0, a) > 0$, $(\partial f/\partial x)(x, a) \leq 0$, $(\partial^2 f/\partial x^2)(x, a) > 0$ *for all* $x \in [1, x_0]$.

(c) $y = f(x, a)$ *is defined globally on* $[1, \infty)$ *and* $f(x, a) > 0$, $(\partial f/\partial x)(x, a) < 0$, $(\partial^2 f/\partial x^2)(x, a) > 0$ *for all* $x \in [1, \infty)$. *If* $a < 0$, *the* x*-axis is the asymptotic line of* $y = f(x, a)$. *If* $a = 0, y = f(x, a)$ *is asymptotic to* $y = c$ *for some* $c \geq 0$.

Proof. Suppose that $y = f(x, a)$ is neither of type (a) nor (b). Since $f(x, a) \cdot (\partial f/\partial x)(x, a) - x$ is never zero, $f(x, a)$ is defined globally on $[1, \infty)$. It follows that $f(x, a) > 0$, $(\partial f/\partial x)(x, a) < 0$, $(\partial^2 f/\partial x^2)(x, a) > 0$ on $[1, \infty)$. Since $y = f(x, a)$ does not intersect with the x-axis, $\lim_{x \to \infty} (\partial f/\partial x)(x, a) = 0$. When $a < 0$, by (2) and $\lim_{x \to \infty} (\partial f/\partial x)(x, a) = 0$, we see that the x-axis

is the asymptotic line of $y = f(x, a)$. When $a = 0$, $y = f(x, a)$ is asymptotic to $y = c$ for some $c \geq 0$.

Proposition 2. *$y = f(x, 0)$ is of type* (a).

Proof. When $a = 0$, (2) becomes

$$\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \frac{dy}{dx} \left(y\frac{dy}{dx} - x\right)^{-1}. \tag{3}$$

First we show that $y = f(x, 0)$ is not of type (b). If there is $x_0 > 1$ such that $(\partial f/\partial x)(x_0, 0) = 0$, then $y = f(x_0, 0)$ and $y = f(x, 0)$ are two distinct solutions of (3) with the same initial condition $y(x_0) = f(x_0, 0)$ and $(dy/dx)(x_0) = 0$. This is a contradiction. Next suppose that $y = f(x, 0)$ is of type (c). Then $f(x, 0) > 0$, $-1 \leq (\partial f/\partial x)(x, 0) < 0$ and $(\partial^2 f/\partial x^2)(x, 0) > 0$ on $[1, \infty)$. By (3)

$$\frac{d^2f}{dx^2} \leq -\frac{1}{x}\left\{1 + \left(\frac{df}{dx}\right)^2\right\} \frac{df}{dx}, \tag{4}$$

where we abbreviate $(\partial f/\partial x)(x, 0) = (df/dx)$, $(\partial^2 f/\partial x^2)(x, 0) = (d^2f/dx^2)$. We denote the solution of

$$\frac{d^2y}{dx^2} = -\frac{1}{x}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \frac{dy}{dx} \tag{5}$$

with the initial condition $y(1) = 1$, $(dy/dx)(1) = -1$ as $y = g(x)$. Then we obtain

$$g(x) = -\frac{1}{\sqrt{2}} \log\left(x + \sqrt{x^2 - 1/2}\right) + \frac{1}{\sqrt{2}} \log\left(1 + \frac{1}{\sqrt{2}}\right) + 1. \tag{6}$$

By the comparison theorem of a first-order ordinary differential equation, we get $df/dx \leq dg/dx$ for all $x \in [1, \infty)$. Integrating this inequality, we get $f(x, 0) \leq g(x)$ for all $x \in [1, \infty)$. Since $\lim_{x\to\infty} g(x) = -\infty$, $\lim_{x\to\infty} f(x, 0) = -\infty$. This is a contradiction. The conclusion follows from Lemma 3.

Proposition 3. *There is $a_0 < 0$ satisfying the following property*:

(i) *$y = f(x, a)$ is of type* (a) *for all $a \in (a_0, 0]$.*
(ii) *$\lim_{a\to a_0+0} x_a = \infty$, where x_a is the x-coordinate of the intersection of $y = f(x, a)$ with the x-axis.*
(iii) *$y = f(x, a_0)$ is of type* (c).

Proof. We set

$$A = \{a \in (-\infty, 0]; y = f(x, \alpha) \text{ is of type (a) for all } \alpha \in [a, 0]\}. \tag{7}$$

By Proposition 2, Lemma 3 and the continuity on the parameter of the solution of an ordinary differential equation, we see that A is a nonempty connected

open subset of $(-\infty, 0]$. Set $A = (a_0, 0]$, where $-\infty \le a_0 < 0$. For every $a \in (a_0, 0]$, $0 \le f(x, a) \le 1$, $-1 \le (\partial f/\partial x)(x, a) < 0$ and $(\partial^2 f/\partial x^2)(x, a) > 0$ on $[1, x_a]$. So we get from (2)

$$\frac{\partial^2 f}{\partial x^2}(x, a) \ge -\frac{a}{4} f(x, a) \qquad \text{on } [1, x_a]. \tag{8}$$

We denote by $h(x, a)$ the solution of $dy^2/dx^2 = -ay/4$ satisfying the initial condition $y(1) = 1$, $(dy/dx)(1) = -1$. We have

$$\begin{aligned} h(x, 0) &= 2 - x, \\ h(x, a) &= \frac{1}{2}\left(1 - \frac{2}{\sqrt{-a}}\right) \exp\left(\frac{\sqrt{-a}(x-1)}{2}\right) \\ &\quad + \frac{1}{2}\left(1 + \frac{2}{\sqrt{-a}}\right) \exp\left(-\frac{\sqrt{-a}(x-1)}{2}\right), \ (a < 0). \end{aligned} \tag{9}$$

When $-4 < a < 0$, $y = h(x, a)$ intersects with the x-axis at $x = (-a)^{-1/2} \cdot \log((2 + \sqrt{-a}/(2 - \sqrt{-a})) + 1$. For $a \in (a_0, 0]$, we set

$$B_a = \{x \in (1, x_a]; f(t, a) > h(t, a), \frac{\partial f}{\partial x}(t, a) > \frac{\partial h}{\partial x}(t, a) \quad \text{for all } t \in (1, x]\}. \tag{10}$$

Since $(\partial^2 f/\partial x^2)(x, a) \ge (-a/2) f(x, a)$ and $(\partial^2 h/\partial x^2)(t, a) = (-a/2) h(x, a)$, we see easily that $\sup B_a = x_a$. Thus for $a \in (a_0, 0]$, $f(x, a) \ge h(x, a)$ on $[1, x_a]$. Therefore $x_a \ge (-a)^{-1/2} \log((2 + \sqrt{-a})/(2 - \sqrt{-a})) + 1$ for all $a \in (a_0, 0)$. Since $\lim_{a \to -4+0} (-a)^{-1/2} \log((2 + \sqrt{-a})/(2 - \sqrt{-a})) + 1 = \infty$, we get $a_0 > -4$ and $\lim_{a \to a_0 + 0} x_a = \infty$. From (i) and (ii) we get (iii).

Theorem. *There is an $O(2) \times O(2)$-invariant complete hypersurface with constant negative scalar curvature in E^4.*

Proof. By Proposition 3, there is $a_0 < 0$ such that $y = f(x, a_0)$ $(x \ge 1)$ is of type (c). Since the equations (1), (2) are symmetric with respect to x and y, we can extend $y = f(x, a_0)$ naturally on $(0, \infty)$ to get a global solution γ of (1). Since γ has an infinite length, M_γ is complete.

Corollary. *There is a complete hypersurface of constant negative scalar curvature in E^n for $n \ge 4$.*

Remarks. (1) By numerical analysis, we see that the scalar curvature of M_γ constructed in the proof of the theorem is about -0.52.

(2) It seems that except $S^3(r)$ and $S^1(r) \times E^2$, M_γ in the theorem is the only $O(2) \times O(2)$-invariant complete hypersurface with constant scalar curvature in E^4 (up to homothety). However, it seems that there are many $O(p) \times O(q)$-invariant complete hypersurfaces with constant scalar curvature in E^{p+q} for $p + q > 4$. These problems will be studied in the future.

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REFERENCES

1. W-Y. Hsiang, *On rotational W-hypersurfaces in spaces of constant curvature and generalized laws of sine and cosine*, Bull. Inst. Math. Acad. Sinica **11** (1983), 349–373.
2. W-Y Hsiang and B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Differential Geometry **5** (1971), 1–38.
3. W-Y. Hsiang, Z-H. Teng and W-C. Yu, *New examples of constant mean curvature immersions of $(2k-1)$-spheres into euclidean $2k$-space*, Ann. of Math. **117** (1983), 609–625.
4. I. Sterling, *A generalization of a theorem of Delaunay to rotational W-hypersurfaces of σ_ℓ-type in H^{n+1} and S^{n+1}*, Pacific J. Math. **127** (1987), 187–197.

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