

DENSE BARRELLED SUBSPACES OF UNCOUNTABLE CODIMENSION

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ABSTRACT. Let E be a Hausdorff barrelled space. If there exists a dense barrelled subspace M such that $(\text{codim}(M) \geq c) \iff [\text{codim}(M) = \dim(E)]$, we say that (M) is a *satisfactory subspace* [11] [E is *barrelledly fit*], respectively. Robertson, Twedde and Yeomans [11] proved that E has a barrelled countable enlargement (BCE) if it has a satisfactory subspace. (Trivially) E has a satisfactory subspace if $\dim(E) \geq c$ and E is barrelledly fit. We show that E is barrelledly fit (and $\dim(E) \geq c$) if $E \neq \emptyset$ and either (i) E is an (LF) -space, or (ii) E is an infinite-dimensional separable space and the continuum hypothesis holds. Conclusion: barrelledly fit spaces and their permanence properties arise from and advance the study of BCE's.

We have shown in [10] that every Hausdorff barrelled space E with $E' \neq E^*$ has a dense denumerable-codimensional subspace, necessarily barrelled by [13]. In [20] it is shown, assuming a "Condition $(2')$ " strictly weaker than the continuum hypothesis, that E must always contain a dense uncountable-codimensional subspace. In fact, a number of papers explicitly involve dense infinite-codimensional subspaces of barrelled spaces, including [4, 5, 21] and others individually cited later. This paper focuses on dense large-codimensional subspaces that, in addition, are barrelled, contrasting with the *non*-barrelled focus of [17].

A Hausdorff topological vector space E is *fit* [18], [20] if it contains a dense subspace M whose codimension is as large as possible, namely, if $\text{codim}(M) = \dim(E)$. Call E *barrelledly fit* if M can be chosen so that M is also barrelled. This forces E to be (locally convex and) barrelled; clearly, E is barrelledly fit if and only if there is a dense barrelled subspace with equal dimension and codimension. A barrelledly fit space of dimension $\geq c$ ($= 2^{\aleph_0}$) has a satisfactory subspace; the converse fails (Example 1). Implicit in [3] is a discussion (p. 246) of satisfactory subspaces and a result (Remark (ii)) on barrelledly fit spaces which is generalized here. We improve several parts of [11] and show by later theorems and examples what are the best possible generalizations, in various different senses, of the Abstract's (i). We do not know the answer to

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the basic question “Is every barrelled, fit space barrelledly fit?”. Some further open questions are discussed.

For standard terminology and notation not herein defined or otherwise referenced, one may refer to [6–9]. We assume that all spaces are locally convex and Hausdorff (with real or complex scalar field).

A space E is called a *(db)-space* if, whenever E is the union of an increasing sequence of subspaces, one of them is dense and barrelled. (See [11] and [14]. Such spaces are also called *suprabarrelled* in [23].) By an oft-used technique, any infinite-dimensional space E is the union of an increasing sequence of subspaces, each with dimension and codimension equal to the dimension of E (see [13], p. 96, [11], p. 107 and [3], p. 246). This yields immediately:

Proposition 1. *Every infinite-dimensional (db)-space is barrelledly fit. (A parallel result: Every infinite-dimensional Baire space is “Baire-ly fit”).*

Example 1. *A space with a satisfactory subspace need not be barrelledly fit:* let F be a space of dimension c with a satisfactory subspace M , and let G be a space of dimension $d > c$ with its strongest locally convex topology. Then $E = F \times G$ has a satisfactory subspace $M \times G$. But E is not barrelledly fit. For if L is a dense subspace of E and π is the projector of E onto G , then $\pi(L)$ is dense in G and so $\pi(L) = G$. Hence $(F \times \{0\}) + L = E$, so $\text{codim}(L) \leq \dim(F) = c < d = \dim(E)$, and E is not fit. \square

Our next Proposition is an immediate consequence of the obvious fact that E (locally convex) is fit if [and only if] $(E, \sigma(E, E'))$ is fit, and the fact proved in [18] that if E has a 0-neighborhood base whose cardinality does not exceed the dimension of E , then E is fit [e.g. every infinite-dimensional metrizable space is fit]. We prefer, however, to give a short independent proof: the proposition plays a vital role in one of our main results.

Proposition 2. *If E has infinite dimension not exceeded by that of E' , then E is fit.*

Proof. Let d be the (infinite) dimension of E , and let \mathcal{S} be a collection of d subspaces each of dimension d such that E is algebraically the direct sum of the members $[d^2 = d$; cf. [10], Example (d)]. For each member G of \mathcal{S} , by the Hahn-Banach Theorem

$$\dim(G') \leq \dim(E') \leq d < 2^d = \dim(G^*)$$

so $G' \neq G^*$ and there is a dense 1-codimensional subspace G_0 of G . It follows that $E_0 = \text{span}(\bigcup_{G \in \mathcal{S}} G_0)$ is a dense d -codimensional subspace of E . \square

Theorem 1. *Assume the continuum hypothesis $[\aleph_1 = c]$. Let E be a barrelled space containing a subspace M with $\dim(M) \geq c \geq \dim(M')$. Then E contains a dense barrelled subspace with codimension equal to $\dim(M)$. In particular, E contains a satisfactory subspace.*

Proof. Let N be an algebraic complement of M in E .

Case I. $\dim(M) = c$. By Proposition 2, there exists a subspace M_0 of M which is dense and c -codimensional in M . Let B be a Hamel basis for an algebraic complement of M_0 in M . There is a collection \mathcal{B} of 2^c subsets of B , each with cardinality c , such that $T_1 \cap T_2$ is countable for distinct T_1, T_2 in \mathcal{B} , by Theorem 1.3, p. 48 of [8]. The proof proceeds by contradiction: suppose that, for each T in \mathcal{B} , $E_T = M_0 + \text{span}(B \setminus T) + N$ is not barrelled, so that there exists a sequence S_T in E' which is pointwise bounded on E_T but not on T ; let R_T be the sequence of restrictions to M . The mapping $T \mapsto R_T$ cannot be one-to-one, since there are only c distinct sequences in $M'[(2^{\aleph_0})^{\aleph_0} = c]$. Let T_1, T_2 be distinct with $R_{T_1} = R_{T_2}$. Then R_{T_i} is pointwise bounded on $L = M_0 + \text{span}(B \setminus (T_1 \cap T_2))$, and so S_{T_i} is pointwise bounded on $L + N$, a dense subspace of E , barrelled by [13]. Therefore, S_{T_i} is $\sigma(E', E)$ -bounded, a contradiction.

Case II. $\dim(M) > c$. For each non- $\sigma(M', M)$ -bounded sequence in M' , choose one point in M at which the sequence is unbounded. The collection D of all these points has cardinality $\leq c$, so $P = \text{span}(D)$ has codimension in M equal to $\dim(M) [> c]$, and any sequence in M' bounded on points of P must be $\sigma(M', M)$ -bounded. Thus the bipolar theorem assures that P is dense in M , and $P + N$ is dense in E , with codimension equal to $\dim(M)$. Moreover, any sequence in E' bounded on points of $P + N$ must be bounded on points of $M + N = E$, so $P + N$ is barrelled. \square

Corollary 1. *Assuming the continuum hypothesis, if $E \not\cong \varphi$ is an infinite-dimensional barrelled space with $\dim(E') \leq c$, then E is barrelledly fit.*

Proof. Isomorphically, φ is the only \aleph_0 -dimensional barrelled (Hausdorff) space; apply the theorem with $M = E$. \square

Corollary 2. *Assuming the continuum hypothesis, every barrelled space which contains a separable subspace of uncountable dimension has a satisfactory subspace. In particular, every infinite-dimensional separable barrelled space except φ (i.e. every separable barrelled space E with $E' \neq E^*$) has a satisfactory subspace; even more, is barrelledly fit.*

Proof. The dual of a separable space has dimension $\leq c$. \square

Assuming the continuum hypothesis, Ian Twedde in [22] constructed a dense barrelled subspace ψ of $\omega [= K^{\aleph_0}$, a scalar field product space], demonstrating several interesting properties of ψ , including barrelledly fitness. This follows immediately from Corollary 2. One could also observe that, without regard to the continuum hypothesis, if $\aleph_0 < d \leq c$ then K^d is a separable barrelled space. Now assume the continuum hypothesis: the Hahn–Banach Theorem and Corollary 1 assure that every infinite-dimensional barrelled subspace of K^d is barrelledly fit, but the latter part of Corollary 2 does not, since K^d has nonseparable barrelled subspaces. For example, the subspace E_0 of members

of K^d having only countably many nonzero coordinates is a dense barrelled nonseparable subspace.

Theorem 5 of [11] states that if E has a satisfactory subspace, then E has a barrelled countable enlargement (BCE). In [19] this result is sharpened by replacing c with \aleph_1 , again assuming “Condition (2’)”. The existence of a BCE is also discussed in [22] and in [1].

Corollary 3. *Assuming the continuum hypothesis, every separable infinite-dimensional barrelled space except φ has a BCE.*

Proof. By Corollary 2 and Theorem 5 of [11]. \square

We shall show later (Proposition 6) that every (LF) -space except φ has a satisfactory subspace, and so a BCE. We may deduce this also here, under the assumption of the continuum hypothesis. For if $E = \varinjlim E_n$, then one of the Fréchet spaces, say E_p , is infinite-dimensional ($E \neq \varphi$). But then E_p contains a separable c -dimensional subspace M , and M remains separable under the coarser topology induced by E , and Corollary 2 applies. In fact, in the case of E an $(LF)_{2 \text{ or } 3}$ -space [14]–[16], E has a separable quotient to which Corollary 3 or even Proposition 1 applies (see [14], Corollary to Theorem 3 and [15], Theorems 3, 4, 1 and 9), and one is then tempted to try some version of [1], Theorem 5 to see that E has a BCE. A different and fruitful approach develops permanence properties of barrelledly fit spaces, adding to the catalogue of spaces having a satisfactory subspace and so a BCE, and yielding three generalizations of the (LF) -space result (Proposition 6, Theorems 3, 4), all without requiring the continuum hypothesis.

Proposition 3. *Arbitrary Cartesian products and locally convex direct sums of barrelledly fit spaces are barrelledly fit.*

Proof. Obvious. \square

Example 2. *Quotients of barrelledly fit spaces need not be barrelledly fit:* let $E \neq \{0\}$ be any barrelledly fit space. Then $E \times \varphi$ is barrelledly fit but has the nonfit φ as a quotient. Considering $\{0\} \times \varphi$ shows that: *Closed barrelled subspaces of barrelledly fit spaces need not be barrelledly fit.* Further, by Example 1: *A space may have a barrelledly fit quotient and a nontrivial barrelledly fit closed subspace, without being barrelledly fit.* \square

The next proposition stands in contrast to the last statement. By [12], the three-space problem is true for $P = \text{barrelled}$. (Here we are using the terminology of [2].) With a short straightforward argument, this yields:

Proposition 4. *The three-space problem is true for $P = \text{barrelledly fit}$.*

Some fit spaces have nonfit countable-codimensional subspaces [18]. However:

Theorem 2. *Every countable-codimensional subspace M of a barrelledly fit space E is barrelledly fit.*

Proof. Let N be an algebraic complement of M and let E_0 be a barrelled dense subspace of E with codimension equal to the dimension of E . Set $M_0 = (E_0 + N) \cap M$ and note that, for $E \neq \{0\}$, $\aleph_0 < \dim(E) = \dim(E) - \aleph_0 = [\text{codim}(E_0) \text{ in } E] - \aleph_0 \leq \text{codim}(E_0 + N) \text{ in } E \leq \text{codim}(M_0) \text{ in } E$, so that, in any case, $\text{codim}(M_0) \text{ in } E = \dim(E)$. Similarly, $\text{codim}(M_0) \text{ in } M = \text{codim}(M_0) \text{ in } E = \dim(E) = \dim(M)$.

Again by the same transfinite arithmetic, if M_1 is a countable-dimensional subspace of M , $M_0 + M_1$ still has codimension in M equal to the dimension of M . Our proof is carried by demonstrating such an M_1 with $M_0 + M_1$ barrelled and dense in M . It is a trivial exercise to show that $M_0 + N$ contains the dense barrelled space E_0 and thus is itself barrelled. By [13], M_0 is barrelled. Again, $F = \overline{M_0} + N$ is barrelled, so that any (necessarily countable-dimensional) algebraic complement of $\overline{M_0}$ in F is a topological complement and has its strongest locally convex topology by [13, p. 92]. F , then, is the topological direct sum of two subspaces of E , one closed and one complete, so F is closed. Hence $F(\supset E_0)$ is all of E . Therefore $\overline{M_0}$ is countable-codimensional in E , and $\overline{M_0} \cap M$ is countable-codimensional in M . But M is barrelled, and, again, any algebraic complement M_1 of $\overline{M_0} \cap M$ in M has its strongest locally convex topology, with $M = (\overline{M_0} \cap M) \oplus M_1$ topologically. Thus the subspace $M_0 + M_1$ is $M_0 \oplus M_1$, the topological direct sum of two barrelled spaces: $M_0 + M_1$ is barrelled and clearly dense in M , completing the proof. \square

Note. The proof also shows: if E has a satisfactory subspace, so does M (cf. [1], Theorem 6).

In discussing inductive limits, we need the following three simple Lemmas.

Lemma 1. *Let (E_1, \mathcal{T}_1) and (E, \mathcal{T}) be two barrelled spaces, with the first continuously included in the second. If V is an absolutely convex closed subset of (E, \mathcal{T}) which absorbs points of a dense barrelled subspace L of (E_1, \mathcal{T}_1) , then V absorbs points of E_1 . Indeed, $V \cap E_1$ is a \mathcal{T}_1 -neighborhood of 0.*

Lemma 2. *With (E_1, \mathcal{T}_1) and (E, \mathcal{T}) as above, if M is a subspace of E such that $M \cap E_1$ is a dense barrelled subspace of (E_1, \mathcal{T}_1) and such that $E_1 + M$ is a dense barrelled subspace of (E, \mathcal{T}) , then M is a dense barrelled subspace of (E, \mathcal{T}) .*

Proof. Let V be any absolutely convex closed subset of (E, \mathcal{T}) which absorbs points of M . Then by Lemma 1 with $L = M \cap E_1$, V absorbs points of E_1 , and hence points of $E_1 + M$, and thus points of E , now applying Lemma 1 with the two spaces coincident. Therefore V is a \mathcal{T} -neighborhood of 0. It follows that M is dense and barrelled in (E, \mathcal{T}) . \square

Lemma 3. *Let (E, \mathcal{T}) be the (Hausdorff) inductive limit of an increasing system $\{(E_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \Gamma}$ of barrelled spaces. That is, Γ is a totally ordered set; each E_α*

is barrelled and for $\alpha, \beta \in \Gamma$ with $\alpha < \beta$, $E_\alpha \subset E_\beta$ and \mathcal{T}_β induces a topology on E_α coarser than \mathcal{T}_α ; $E = \bigcup_{\alpha \in \Gamma} E_\alpha$; and \mathcal{T} is a Hausdorff locally convex topology, the finest one which induces on E_α a topology coarser than \mathcal{T}_α , for each α . If M is a subspace of E such that, for all α in Γ , $M \cap E_\alpha$ is a dense barrelled subspace of $(E_\alpha, \mathcal{T}_\alpha)$, then M is a dense barrelled subspace of (E, \mathcal{T}) .

Proof. Let V be any absolutely convex closed subset of (E, \mathcal{T}) which absorbs points of M . Then, for each α , $V \cap E_\alpha$ is a \mathcal{T}_α -neighborhood of 0 by Lemma 1, so V is a \mathcal{T} -neighborhood of 0. \square

The remaining propositions and theorems generalize Note 3.2, its Corollary and Corollary 3 to Lemma 3 in [11].

Proposition 5. *Let (E_1, \mathcal{T}_1) be barrelled and continuously included in the barrelled space (E, \mathcal{T}) . If E_0 is a dense barrelled subspace of (E_1, \mathcal{T}_1) and N is any algebraic complement of E_1 in E , then $E_0 + N$ is a dense barrelled subspace of (E, \mathcal{T}) . In particular, if (E_1, \mathcal{T}_1) has a satisfactory subspace, so does E .*

Proof. Let $M = E_0 + N$. Then $E = E_1 + N = E_1 + M$ and $E_0 = M \cap E_1$, so Lemma 2 applies. Also, the codimension of E_0 in E_1 equals that of $E_0 + N$ in E . \square

Proposition 6. *Let (E, \mathcal{T}) be the (Hausdorff) inductive limit of an increasing system $\{(E_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \Gamma}$ of barrelled spaces. If some $(E_\alpha, \mathcal{T}_\alpha)$ has a satisfactory subspace, then so does E . In particular, every (LF)-space except ϕ has a satisfactory subspace and a BCE.*

Proof. In general, Proposition 5 applies. In particular, any (LF)-space non-isomorphic to ϕ continuously includes an infinite-dimensional Fréchet space E_1 . Thus $\dim(E_1) \geq c$ and E_1 has a satisfactory subspace by Proposition 1. \square

Theorem 3. *The (Hausdorff) inductive limit (E, \mathcal{T}) of an increasing sequence $\{(E_n, \mathcal{T}_n)\}_{n=1}^\infty$ of barrelledly fit spaces is barrelledly fit.*

Proof. Let $E_{n,d}$ be a dense barrelled subspace of (E_n, \mathcal{T}_n) whose dimension and codimension in E_n are equal ($n = 1, 2, \dots$). Set $F_1 = E_{1,d} = M_1$ and let $H_1 = G_1$ be any algebraic complement of F_1 in E_1 . Then $\dim(F_1) = \dim(H_1) = \dim(E_1)$. Let M_2 be any algebraic complement of $E_1 \cap E_{2,d}$ in $E_{2,d}$. Then $E_1 + M_2 \supset E_{2,d}$. Let G_2 be any algebraic complement of $E_1 + M_2$ in E_2 , and set

$$F_2 = F_1 + M_2, \quad H_2 = G_1 + G_2 = H_1 + G_2$$

so that F_2 and H_2 are algebraically complementary in E_2 and $F_1 \subset F_2, H_1 \subset H_2$. If $\dim(E_2) = \dim(E_1)$, we clearly have $\dim(F_2) = \dim(H_2) = \dim(E_2)$, and this result still holds by elementary transfinite arithmetic in case $\dim(E_2) > \dim(E_1)$. By Lemma 2, F_2 is a dense barrelled subspace of (E_2, \mathcal{T}_2) . Using

induction, we can continue the process to obtain increasing sequences $\{F_n\}_{n=1}^\infty$, $\{H_n\}_{n=1}^\infty$ such that F_n is a dense barrelled subspace of (E_n, \mathcal{T}_n) and H_n is an algebraic complement with $\dim(F_n) = \dim(H_n) = \dim(E_n)$ ($n = 1, 2, \dots$). Set

$$F = \bigcup_{n=1}^\infty F_n, \quad H = \bigcup_{n=1}^\infty H_n.$$

It is clear that F and H are algebraically complementary in E , $\dim(F) = \dim(H) = \dim(E)$, and F is a dense barrelled subspace by Lemma 3. \square

Example 3. *There are strict inductive limits of (uncountable) increasing systems of barrelledly fit spaces that are not barrelledly fit.* Let $E \neq \{0\}$ be any barrelledly fit space, let μ be the first ordinal whose cardinality exceeds the dimension of E , and let F be a vector space with Hamel basis $\{x_\alpha\}_{\alpha \in \mu}$ indexed by μ . For each $\alpha \in \mu$, set $F_\alpha = \text{span}(\{x_\beta\}_{\beta \leq \alpha})$, given its strongest locally convex topology, and set $G_\alpha = E \times F_\alpha$. Then $\{G_\alpha\}_{\alpha \in \mu}$ is an increasing system of barrelledly fit spaces [$\dim(F_\alpha) \leq \dim(E)$] whose inductive limit is strict and is equal to $E \times F$, where F is given its strongest locally convex topology. By the argument of Example 1, $E \times F$ is not fit. \square

Analogues to some of these results, including Theorem 3, Example 3, and the following Theorem 4, are found in [18], with “fit” replacing “barrelledly fit”.

Theorem 4. *The (Hausdorff) inductive limit of an arbitrary increasing system of infinite-dimensional (db)-spaces is barrelledly fit.*

Proof. Let (E, \mathcal{T}) and $\{(E_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \Gamma}$ be as in Lemma 3 with each $(E_\alpha, \mathcal{T}_\alpha)$ an infinite-dimensional (db)-space. We assume that Γ has no countable cofinal subset, since the countable case is covered by Theorem 3. Let \mathcal{G} be the collection of all quadruples (S, F, H, g) such that: $\emptyset \neq S \subset \Gamma$; F is a dense barrelled subspace of L_S , the inductive limit of the increasing system $\{(E_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in S}$, with algebraic complement H (in L_S); and g is a one-to-one linear map from F onto H . By Proposition 1, \mathcal{G} is nonempty. We partially order \mathcal{G} by writing $(S_1, F_1, H_1, g_1) \leq (S_2, F_2, H_2, g_2)$ if (and only if) $S_1 \subset S_2$, $F_1 \subset F_2$, $H_1 \subset H_2$, and $g_1 = g_2|_{F_1}$. Let \mathcal{C} be a chain in \mathcal{G} . Then applying Lemma 3 to the inductive limit of all the spaces L_S with $(S, F, H, g) \in \mathcal{C}$, it follows that \mathcal{C} has a least upper bound. By Zorn's Lemma, there is a maximal element (S_0, F_0, H_0, g_0) in \mathcal{G} . Now suppose there is some α in Γ such that $L_{S_0} = F_0 + H_0$ is an infinite-codimensional subspace of E_α . Let B be a basis of an algebraic complement of L_{S_0} in E_α and partition B into denumerably many subsets each having the same cardinality as B . Applying the technique used in Proposition 1 to the (db)-space $E_\alpha = L_{S_0} + \text{span}(B)$ yields algebraically complementary subspaces M and N in $\text{span}(B)$ of equal dimension such that $L_{S_0} + M$ is dense and barrelled in $(E_\alpha, \mathcal{T}_\alpha)$. By equality of dimension, there is a one-to-one linear extension g_1 of g_0 mapping $F_0 + M$ onto $H_0 + N$. Moreover, $F_0 + M$ is a dense barrelled subspace of $(E_\alpha, \mathcal{T}_\alpha)$

by Lemma 2, and the inductive limit space $L_{S_0 \cup \{\alpha\}}$ is just $(E_\alpha, \mathcal{T}_\alpha)$, so that $(S_0 \cup \{\alpha\}, F_0 + M, H_0 + N, g_1)$ is in \mathcal{G} , contradicting maximality. Thus it is false to suppose that L_{S_0} is infinite-codimensional in any E_α . It now follows that L_{S_0} must be finite-codimensional in E itself: otherwise there would be a strictly increasing sequence $\{\alpha_n\}_{n=1}^\infty \subset \Gamma$ such that $L_{S_0} \subset E_{\alpha_1} \subset E_{\alpha_2} \subset \dots$, with each subspace of (finite) positive codimension in the succeeding one. But as $\{\alpha_n\}_{n=1}^\infty$ is not cofinal, there is some α in Γ which exceeds each α_n , thus forcing the contradiction that L_{S_0} is infinite-codimensional in E_α . Let N be a (finite-dimensional) algebraic complement of L_{S_0} in E . By Proposition 5, $F_0 + N$ is a dense barrelled subspace of E , and clearly, $\text{codim}(F_0 + N)$ in $E = \dim(H_0) = \dim(F_0) = \dim(F_0 + N)$. \square

Example 4. *There are strict inductive limits of directed systems [7] of infinite-dimensional (db)-spaces that are not barrelledly fit:* specialize Example 1 by taking F to be a (db)-space. Then $E = F \times G$ is the strict inductive limit of the (by inclusion) directed system $\{F \times N: N \text{ is a finite-dimensional subspace of } G\}$ of infinite-dimensional (db)-spaces which, as was shown, fails to be fit. \square

Open questions. In this paper we have answered affirmatively in special cases only the basic, still open question “Does every barrelled space E with $E' \neq E^*$ have a BCE?”. Assuming the generalized continuum hypothesis, [20] shows that E (as above) always contains a barrelled, fit subspace of dimension $\geq c$; thus if every barrelled, fit space is barrelledly fit, then E has a satisfactory subspace and a BCE. In [11] the substitute question was (is) “Does every E (as above) have a satisfactory subspace?”; in this paper, “If E is barrelled and fit, is E barrelledly fit?”. All three of the questions seem intrinsically interesting, and as suggested by the reasoning above and in [11], “Yes” to either one of the latter two questions would imply “Yes” to the preceding one(s).

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