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# THE IMAGE OF $H_*(BSU; Z_p)$ IN $H_*(BU; Z_p)$

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ABSTRACT. In this note we construct explicit polynomial generators for the image of $H_*(BSU; Z_p)$ inside $H_*(BU; Z_p)$.

## INTRODUCTION

Recently various families of generators of the image of $H_*(BSO; Z_2)$ inside $H_*(BO; Z_2)$, have been constructed by Bahari [1], Baker [2], Kochman [4, 5], Papastavridis [7] and Pengelley [8, 9]. In this note we find relatively simple polynomial generators for the image of $H_*(BSU; Z_p)$ inside $H_*(BU; Z_p)$.

It is well known that

$$H_*(BU; Z_p) = Z_p[x_1, x_2, \dots, x_n, \dots],$$

where $p$ is a prime number and $x_1, x_2, \dots, x_n, \dots$ are the well-known elements of $H_*(BU; Z_p)$, (see, for example, [2, Proposition 10]).

If $n$ is not a power of $p$, then nonnegative integers $a$, $b$, $r$, such that $n = ap^{r_1} + bp^r$ and $0 < b < p$, are uniquely defined. We put $t = p^r$. Then we define

$$y_n = \sum_{i=0}^{t} \sum_{j=0}^{n-2t} (-1)^{i+j} \frac{n-2i-j}{t-i} \binom{n-t-1-i-j}{t-1-i} x_i x_j x_{n-i-j}.$$

*Remark.* The symbol $\binom{i}{j}$ is the binomial coefficient. By definition we put

$$\frac{a}{0}\binom{b}{-1} = 1 \quad \text{for } a \geq 0 \text{ and } b \geq -1.$$

If $n = p^r$ and $r > 0$, then we define $y_n = (x_{n/p})^p$.

In this note, we will prove the following theorem.

**Theorem 1.** *The image of $H_*(BSU; Z_p)$ in $H_*(BU; Z_p)$, under the obvious monomorphism, is the polynomial algebra generated by $y_2, y_3, \dots, y_n, \dots$.*

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*Proof of Theorem* 1. From B. Gray's paper, [3] we will need the following lemma.

**Lemma 2.** *Let* $d\colon H_*(BU;Z_p) \to H_*(BU;Z_p)$ *be the derivation defined by* $d(x_n) = x_{n-1}$. *Then the sequence*

$$0 \to H_*(BSU;Z_p) \to H_*(BU;Z_p) \xrightarrow{d} H_*(BU;Z_p) \to 0$$

*is exact.*

*Proof.* See [1, Proposition 4.2]. The proof refers to $BO$, but it works the same for $BU$.

It is easy to observe, that if $n$ is not a power of $p$, then $y_n$ is indecomposable in $H_*(BU;Z_p)$. The reason is the following: Let $n = ap^{r+1} + bp^r$, with $r \geq 0$ and $0 < b < p$. Then the coefficient of $x_n$ in the expression for $y_n$ is

$$\frac{n}{t}\binom{n-t-1}{t-1} = (ap+b)\cdot\binom{ap^{r+1}+(b-1)p^r-1}{p^r-1} \equiv b \not\equiv 0 \mod p.$$

(We use the well-known formula which computes the binomial coefficient mod $p$.) From this remark it follows that the elements $y_2, y_3, \dots, y_n, \dots$ are polynomially independent. Furthermore, it is obvious, that the graded polynomial algebra $Z_p[y_2, y_3, \dots, y_n, \dots]$ has in each degree the same $Z_p$-dimension as $H_*(BSU;Z_p)$. So, Lemma 2 tells us that, in order to prove Theorem 1, it is enough to prove that the $y_n$'s belong to the kernel of the derivation $d$. The case where $n$ is a power of $p$ presents no problem because in this case $y_n$ is a $p$ th power. Our next lemma is all we need to settle the case where $n$ is not a power of $p$.

**Lemma 3.** *Let us define*

$$c_{i,j} = \begin{cases} 1, & \text{if } i = 0 \text{ and } j \geq -1. \\ 2\binom{j}{i-1} + \binom{j}{i} = \frac{j+i+1}{i}\binom{j}{i-1}, & \text{if } i \geq 1 \text{ and } j \geq i-1. \end{cases}$$

*Then* $c_{i+1,j+1} - c_{i,j} - c_{i+1,j} = 0$ *for* $i \geq 0$ *and* $j \geq i-1$.

*Proof.* It is obvious. Just direct calculations.

Now, with this lemma in our hand, it is an easy matter to prove that $y_n$ belongs to the kernel of the derivation $d$ for the case where $n$ is not a power of $p$. We calculate $d(y_n)$, using the usual rule of the derivation of a product, and our last lemma is all we need to see that we get zero.

*Comment.* The referee and the editor, H. Miller, suggested a comment on the discovery of the formula for $y_n$. Each $y_n$ is a linear combination of monomials of length $\leq 3$. The problem is to guess the coefficients of this linear combination. The double summation covers a rectangle in the $i-j$ plane. The edges of the rectangle ensure the appearance of the indecomposable monomial $x_n = x_0x_0x_n$. The coefficients are combinations of binomial coefficients designed to produce cancellations in $d(y_n)$, according to the Pascal triangle.

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