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TWO CLASSES OF FRÉCHET-URYSOHN SPACES

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ABSTRACT. Arhangel'skii introduced five classes of spaces, α_i-spaces $(i < 5)$, which are important in the study of products of Fréchet-Urysohn spaces. For each $i < 5$, each α_i-space is an α_{i+1}-space and it follows from the continuum hypothesis that there are countable α_{i+1}-spaces which are not α_i-spaces. A v-space (w-space) is a Fréchet-Urysohn α_1-space (α_2-space). We show that there is a model of set theory in which each α_2-space (w-space) is an α_1-space (v-space).

INTRODUCTION

Arhangel'skii defines a point $x \in X$ to be an *α_1-point* (*α_2-point*) if whenever F_n is a sequence converging to x, for each $n < \omega$, there is a sequence F converging to x such that $F_n - F$ is finite ($F_n \cap F$ is infinite) for each $n < \omega$. Furthermore a point is called an α_0-point if it has a countable neighbourhood base. A space is called an *α_1-space* if each point is an α_i-point. A space is *Fréchet-Urysohn* if whenever a point is in the closure of a set there is a sequence from it converging to the point.

Nyikos has shown that there is a countable w-space which is not first countable [Ny1, Ny2]. In [Ny2] Nyikos asks if there is a countable w-space which is not a v-space and a countable v-space which is not first-countable. Nyikos [Ny2] produces examples of countable w-spaces which are not v-spaces from a special set-theoretic assumption following from, for example, Martin's Axiom (even $\underline{b} < \underline{d}$). Nogura [N] has shown that MA (even $\underline{b} = \underline{d}$) implies there is an example of a countable v-space which is not first countable. It is shown in [DS] that it is consistent that each countable v-space is first-countable.

Gruenhage [G] introduced w-spaces and Sharma [Sh] obtained their characterization in terms of α_2-spaces. The term v-space seems to be due to Nyikos.

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ω-SPLITTING FAMILIES FROM α_2-POINTS

1. **Definition.** We say that $\mathfrak{X} \subset [\omega]^\omega$ is *ω-splitting* if for each countable family $\{A_n : n \in \omega\} \subset [\omega]^\omega$ there is an $X \in \mathfrak{X}$ such that $|A_n \cap X| = |A_n - X|$ for each $n \in \omega$. We shall say that X is *M-splitting* if $|A \cap X| = |A - X|$ for each $A \in M \cap [\omega]^\omega$.

Let x be a point in a space Y and suppose $F_n \in [Y]^\omega$ is a sequence converging to x for each $n \in \omega$. Identify $\bigcup F_n$ with ω and define $\mathfrak{X}$ to be the set of X which "think" that x is an α_1-point. That is, define $\mathfrak{X}(x, \langle F_n \rangle) = \{X \in [\omega]^\omega : \exists F \subset Y \ (F$ converges to x and $(X \cap F_n) - F$ is finite for each $n \in \omega)\}$. The definition of $\mathfrak{X}(x, \langle F_n \rangle)$ of course depends on the identification of $\bigcup F_n$ with ω, but only up to a permutation on ω and this will never matter to us.

2. **Lemma.** *If $F_n \in [Y]^\omega$ $(n \in \omega)$ converges to an α_2-point x, then $\mathfrak{X} = \mathfrak{X}(x, \langle F_n \rangle)$ is an ω-splitting family.*

Proof. Assume that $\bigcup F_n$ is identified with ω and let $\{A_k : k \in \omega\} \subset [\omega]^\omega$. Let $I = \{j \in \omega : A_j \cap F_n$ is finite for each $n\}$. Choose for each n, a finite subset H_n of $F_n - \bigcup_{m<n} F_m$ so that $A_j - H$ is finite for each $j \in I$ where $H = \bigcup H_n$. Now for each $j \in \omega - I$, choose $n_j \in \omega$ so that $A_j \cap F_{n_j}$ is infinite. Next choose an infinite $B_j \subset A_j \cap F_{n_j}$ so that $B_j \cap B_k = \varnothing$ for $j \neq k$ in $\omega - I$. Now since x is an α_2-point, there is a sequence F converging to x such that $F \cap B_j$ is infinite for each $j \in \omega - I$. It follows that $F \cup H$ hits each A_j in an infinite set and that $F \cup H \in \mathfrak{X}$. Finally any infinite subset of $F \cup H$ is a member of $\mathfrak{X}$ hence there is an $X \in \mathfrak{X}$ which splits $\{A_j : j \in \omega\}$.

3. **Lemma.** *If every ω-splitting family contains an ω-splitting family of cardinality less than $\underline{b}$, then each α_2-point is an α_1-point.*

Proof. Let x be an α_2-point of a space Y and assume F_n is a sequence converging to x for each $n \in \omega$. Since we wish to show that x is an α_1-point we may assume that the F_n's are pairwise disjoint and that $\bigcup F_n = \omega$. Let $\mathfrak{X} = \mathfrak{X}(x, \langle F_n \rangle)$ be defined as above. By Lemma 2 and the hypothesis of this lemma, there is an ω-splitting family $\mathfrak{X}' \in [\mathfrak{X}]^{<\underline{b}}$. For each $A \in \mathfrak{X}'$, choose (by the definition of $\mathfrak{X}$) a sequence F_A converging to x and $f_A \in {}^\omega\omega$ so that $(A \cap F_n) - F_A \subset f_A(n)$ for each $n \in \omega$. Now choose $f \in {}^\omega\omega$ so that $f_A <^* f$ for each $A \in \mathfrak{X}'$ which we may do since $|\mathfrak{X}'| < \underline{b}$. We claim that $F = \bigcup F_n - f(n)$ converges to x, which would show that x is an α_1-point. Indeed, assume F does not converge to x and choose $F' \in [F]^\omega$ such that x is not a limit point of F'. Since $\mathfrak{X}'$ is splitting, choose $A \in \mathfrak{X}'$ so that $A \cap F'$ is infinite. However this contradicts that F_A converges to x since $A \cap F' - F_A$ is finite.

4. *Remark.* We could replace the notion of ω-splitting in Lemma 3 by what one might call " ω-hitting".

ω-SPLITTING FAMILIES AND LAVER FORCING

5. **Theorem.** *In a model obtained by adding ω_2-Laver reals to a model of* CH, *every ω-splitting family contains an ω-splitting family of cardinality ω_1. Hence in this model each α_2-space (w-space) is an α_1-space (v-space).*

The theorem follows from the following four results. Proposition 6 is a collection of standard facts about Laver forcing, Lemma 7 is a standard reflection argument, Lemma 8 is a special case of a general preservation scheme proven in [S2] and Lemma 9 is new. Recall that $T \in \mathbb{L}$ (the Laver poset defined in [L]) if $T \subset {}^{<\omega}\omega$ has a root $t_0 = \mathrm{root}(T)$ and for $t_0 \leq t \in T$ $\{n: t^\frown n \in T\}$ is infinite. $\mathbb{L}$ is ordered by inclusion. For $\lambda \leq \omega_2$ let $\mathbb{P}_\lambda$ be the countable support λ-stage iteration of the forcing notion $\mathbb{L}$.

6. **Proposition.** [CH] *$\mathbb{P}_{\omega_2}$ is an $\omega_2 - cc$ proper poset such that $1 \Vdash_{\mathbb{P}_\lambda} \underline{b} = \underline{c} = |\lambda \cdot \omega_1|^V$. Furthermore if $\mu < \lambda$ then $\mathbb{P}_\lambda$ is forcing isomorphic to $\mathbb{P}_\mu * \mathbb{P}_\lambda$.*

For proofs of the various assertions in Proposition 6 we refer the reader to [L] and [S1].

7. **Lemma.** [CH] *Let $\{X_\alpha : \alpha < \omega_2\}$ be $\mathbb{P}_{\omega_2}$-names such that $1 \Vdash \{X_\alpha : \alpha < \omega_2\} \subset [\omega]^\omega$ is ω-splitting, then there is a $\lambda < \omega_2$ such that $1 \Vdash_{\mathbb{P}_\lambda} \{X_\alpha : \alpha < \lambda\}$ is ω-splitting.*

Notation. If p is a member of a poset P and M is a set, "$p \Vdash_P X$ is M-splitting" will abbreviate $p \Vdash |A \cap X| = |A - X|$ for each P-name $A \in M$ such that $p \Vdash A \in [\omega]^\omega$.

We shall say that a poset P is *ω-splitting* if the following are satisfied: whenever $P \in M$, where M is a countable elementary submodel of $H(\theta)$ for any sufficiently large θ, $p \in M \cap P$ and X is M-splitting then there is some $q < p$ which is (M, P)-generic and such that $q \Vdash X$ is M-splitting.

Note that an iteration of finitely many ω-splitting posets is again ω-splitting; hence proper.

8. **Lemma.** *If $P_\delta = \langle (P_\alpha, Q_\alpha) : \alpha < \delta \rangle \in M$ is a countable support iteration of ω-splitting (hence proper) posets then P_δ is also ω-splitting.*

9. **Lemma.** *Let M be a countable elementary submodel of $H(\omega_3)$ and let $T \in \mathbb{L} \cap M$, then if X is M-splitting there is an $(M, \mathbb{L})$-generic $T' < T$ such that $T' \Vdash X$ is M-splitting. Therefore $\mathbb{L}$ is ω-splitting.*

It may be worthwhile to record the following corollary to the above results.

10. **Corollary.** *If a family is ω-splitting then it will still be ω-splitting after forcing with the countable support iteration of Laver forcing. Furthermore in any*

model obtained by adding iteratively ω_2-Laver reals the splitting number, $\underline{s}$, will be ω_1.

Before proving Lemmas 7–9 let us indicate how Theorem 5 now follows. Let G be $\mathbf{P}_{\omega_2}$-generic over V (a model of CH). Let $\mathfrak{X} = \{X_\alpha : \alpha < \omega_2 = \underline{c}\}$ be an ω-splitting family. Choose $\mathbf{P}_{\omega_2}$-names $\{X_\alpha : \alpha < \omega_2\} \in V$ so that $1 \Vdash \mathfrak{X} = \{X_\alpha : \alpha < \omega_2\}$. By Lemma 7, there is a $\lambda < \omega_2$ so that $1 \Vdash_{P_\lambda} \{X_\alpha : \alpha < \lambda\}$ is ω-splitting. Let $G_\lambda = G \cap \mathbf{P}_\lambda$; hence $V[G_\lambda] \models \{X_\alpha : \alpha < \lambda\}$ is ω-splitting. Since $\mathbf{P}_{\omega_2} \cong \mathbf{P}_\lambda * \mathbf{P}_{\omega_2}$, $V[G] \models \{X_\alpha : \alpha < \lambda\}$ is ω-splitting by Lemmas 8 and 9.

It remains to prove the Lemmas.

Proof of Lemma 7. By Proposition 6, we may assume that for each $\alpha < \omega_2$, there is an $f(\alpha) < \omega_2$ such that X_α is a $\mathbb{P}_{f(\alpha)}$-name. Also if G is $\mathbb{P}_{\omega_2}$-generic, $V[G] \models$ for each $\alpha < \omega_2$, there is a $g(\alpha) < \omega_2$ such that for each set $M \in [[\omega]^\omega]^\omega \cap V[G_\alpha]$ there is an $X \in \{\mathrm{val}(X_\beta, G) : \beta < g(\alpha)\}$ which is M-splitting (since $V[G_\alpha] \models \underline{c} = \omega_1$). Since $\mathbf{P}_{\omega_2}$ is $\omega_2 - cc$ we may assume that $g \in V$. Now let h be a continuous strictly increasing function from ω_2 into ω_2 such that $f(\alpha) + g(\alpha) < h(\alpha + 1)$ for all $\alpha < \omega_2$. Choose $\lambda < \omega_2$ such that $h(\lambda) = \lambda$; it follows that $1 \Vdash_{\mathbf{P}_\lambda} \{X_\alpha : \alpha < \lambda\}$ is ω-splitting.

Proof of Lemma 8. Technically we make the inductive assumption that for each $\beta < \alpha < \delta$ and each $p \in P_\beta$ we have that $p \Vdash P_\alpha / P_\beta$ is ω-splitting; where, as usual, P_α / P_β denotes the P_β-name satisfying the equation $P_\alpha = P_\beta * (P_\alpha / P_\beta)$. However in proving the inductive step we can just force with P_β and work in the extension. Therefore we may make the above inductive assumption and complete the proof by showing that P_δ is ω-splitting. As remarked when we defined the notion of an ω-splitting poset the proof is trivial in case δ is a successor ordinal. Now let θ be a large enough cardinal (i.e. $|\mathscr{P}(P_\delta)| < \theta$) and let $P_\delta \in M$ where M is a countable elementary submodel of $H(\theta)$. Suppose further that $p \in M \cap P_\delta$ and that X is M-splitting. Let $\{A_n : n \in \omega\}$ index the set of P_δ-names of subsets of ω which are in M. We may as well assume that $\mathrm{cf}(\delta) = \omega$ since we will be choosing q to be M-generic and this q will force that the A_n's are essentially $P_{M \cap \delta}$-names. We may therefore choose a countable increasing sequence of ordinals cofinal in δ and by our inductive assumption we may as well assume that $\delta = \omega$.

As in [S2], we shall choose sequences p_n, q_n, k_n and m_n, by induction on n, so that:

(1) $p_n \in P_\omega$, $q_n \in P_n$ and k_n, m_n are P_ω-names of integers,
(2) $p_{n+1}|n = p_n|n$, $p_{n+1} < p_n$, $q_{n+1}|n = q_n$ and $q_n < p_n|n$,
(3) for $n > 0$, $q_n \wedge p_n \Vdash A_n$ is finite or $m_n \in X \cap A_n$ and $k_n \in A_n - X$, ($q_n \wedge p_n$ is meant to denote the element $(q_n \wedge p_n|n)^\wedge p|[n, \omega) \in P_\omega$)
(4) q_n is (M, P_n)-generic and $q_n \Vdash_{P_n} X$ is M-splitting, and
(5) $q_n \Vdash p_n \in M$.

Before we begin the induction, let us comment on what (5) means. It is not the case that p_n will be in M but $q_n \Vdash (\exists p' \in M)$ such that $p'/P_n = p_n/P_n$;

where p/P_n denotes the P_ω/P_n-name corresponding to p in the extension by P_n. It is not even the case that we can bring the "$\exists$" sign outside the forcing statement. However condition 5 is essential in order for the induction to continue. Let $p_1 = p$ where $p \in P_\omega \cap M$ is as chosen above. Since P_1 is proper (and we have chosen θ large enough) we may choose $q_1 < p_1|1$ to be (M, P_1)-generic such that $q_1 \Vdash X$ is M-splitting. Suppose that p_n and q_n have been chosen. Let G_n be P_n-generic such that q_n and $p_n|n$ are in G_n. In $V[G_n]$, let p'_n be the element of M so that $p'_n/P_n = p_n/P_n$, and let $B_{n+1} = \{m\colon (\exists p \in P_\omega/P_n)\ p < p'_n/P_n \text{ and } p \Vdash m \in A_{n+1}\}$. Here we are assuming that A_n/P_n is the P_ω/P_n-name which results from evaluating the P_ω-name A_n and similarly for p'_n. Now $M[G_n]$ is an elementary submodel of $H(\theta)^{V[G_n]}$ which is a model of ZF-P (see [S1]) and $p'_n \in M$, hence $B_{n+1} \in M[G_n]$. Since X is $M[G_n]$-splitting we may choose $m_{n+1} \in X \cap B_{n+1}$ and $p'/P_n < p'_n/P_n$, $p' \in M$ (by elementarity) such that $p' \Vdash m_{n+1} \in A_{n+1}$. Similarly we may choose $p'_{n+1}/P_n < p'/P_n$ and $k_{n+1} \notin X$ such that $p'_{n+1} \in M$, $p'_{n+1}|n \in G_n$ and $p'_{n+1} \Vdash k_{n+1} \in A_{n+1}$. By assumption Q_n is ω-splitting (in $V[G_n]$) hence we may choose $q'_{n+1} < p'_{n+1}(n)$, $q'_{n+1} \in Q_n$ so that q'_{n+1} is $(M[G_n], Q_n)$-generic and so that $q'_{n+1} \Vdash X$ is $M[G_n]$-splitting. Now we use the maximality principle to choose q_{n+1}, p_{n+1} and the names m_{n+1} and k_{n+1} so as to satisfy (1)–(5). That is, $p_n|n \Vdash$ if X is $M[G_n]$-splitting and $M[G_n]$ is an elementary submodel of $H(\theta)^{V[G_n]}$ then there are $p'_{n+1}/P_n \in M[G_n]$, k_{n+1} and m_{n+1} as above. So we may choose a P_n-name, say p''_{n+1}, of an element of P_ω/P_n and P_ω-names k_{n+1} and m_{n+1} so that $p|n \Vdash$ if X is $M[G_n]$-splitting and $M[G_n]$ is an elementary submodel of $H(\theta)^{V[G_n]}$ then $p''_{n+1} \in M[G_n]$ and $p''_{n+1} \Vdash k_{n+1}$ and m_{n+1} are as above. Next we choose a P_n-name q_{n+1} for q'_{n+1}. We let $p_{n+1} = p_n \wedge p''_{n+1}$ and $q_{n+1} = q_n \wedge q_{n+1}$. It is clear that (1), (2) are satisfied. By [S1], q_{n+1} is (M, P_{n+1})-generic and clearly $q_{n+1} \Vdash X$ is M-splitting; hence (4) holds. The reason that (3), (5) are satisfied is that $q_{n+1} \Vdash M[G_{n+1}]$ is an elementary submodel of $H(\theta)^{V[G_{n+1}]}$, hence p''_{n+1} has the desired properties.

Now if $q \in P_\omega$ is such that $q|n = q_n$ for each $n \in \omega$ then q is (M, P_ω)-generic (see [S1]) and $q < p_n$ for each $n \in \omega$. It follows that $q \Vdash X$ is M-splitting, since for each $i < j < \omega$ there is an $n < \omega$ such that $1 \Vdash A_n = A_i - j$ and so $q < q|n \wedge p_n \Vdash m_n \in X \cap A_i - j$ and $k_n \in A_i - (X \cup j)$.

Proof of Lemma 9. Let $T \in L \cap M$ where M is a countable elementary submodel of $H(\omega_s)$ and assume X is M-splitting. Fix indexings $\{A_n | n \in \omega\}$ of $M \cap \{A | A$ is an L-name and $1 \models A \in [\omega]^\omega\}$, and $\{D_n | n \in \omega\}$ of $M \cap \{D \subseteq \mathbf{L} | D$ is dense open$\}$. We shall inductively define a descending sequence $\{T_n | n \in \omega\} \subset \mathbf{L}$ (with $T_0 = T$) so that, for each $n \in \omega$:

1. $T_n \cap^n \omega = T_{n+1} \cap^n \omega$, and
2. If $T' < T_n$ then there is a $t \in T'$ such that
   (a) $(T_n)_t \in M \cap D_n$ and

(b) $(T_n)_t \models X \cap A_n \neq \varnothing$ and $A_n - X \neq \varnothing$.

If we accomplish this, then condition 1 guarantees that $T' = \cap_n T_n \in L$. It is easy to see that condition 2(b) guarantees that $T' \models X$ is M-splitting. Condition 2(a) actually has the double role of ensuring that T' is M-generic and allowing the induction to continue.

Following [L], if S', $S \in \mathbf{L}$ then we use $S' <^0 S$ to denote the situation where $S' < S$ and they have the same root. One of the key facts about Laver forcing from [L] is that if $S \in \mathbf{L}$ and φ is any sentence of the forcing language then there is an $S' <^0 X$ such that either $S' \models \varphi$ or $S' \models \neg\varphi$. Therefore

$$(*) = \begin{cases} \text{if } A \text{ is an } L\text{-name and } F \text{ is a finite set such that} \\ S \models F \cap A \neq \varnothing, \text{ then there is an } S' <^0 S \text{ and an } x \in F \\ \text{such that } S \models x \in A. \end{cases}$$

Let us assume that $0 < n \in \omega$ and that T_{n-1} has been chosen as above. Let I be the set of members of $T_{n-1} -^{\leq n} \omega$ which are minimal with respect to the property that $(T_{n-1})_t \in M$. Note that the minimality of the members of I and condition 2(a) guarantee that the collection $\{(T_{n-1})_t | t \in I\}$ is an antichain in $\mathbf{L}$ which is maximal below T_{n-1}. Now if we find, for each $t \in I$, a condition $T'_t <^0 (T_{n-1})_t$ satisfying condition 2, then we can define T_n to be $\bigcup\{T'_t | t \in I\}$. This works since $\{T'_t | t \in I\}$ is a maximal-below-T_n antichain. For the same reason, repeated uses of Facts 1 to 3 finish the proof.

Fact 1. If $S \in \mathbf{L} \cap M$ and $n \in \omega$, then there is an $S' <^0 X$ such that the collection $(S')_t | (S')_t \in D_n \cap M\}$ is predense below S'.

Fact 2. If $S \in \mathbf{L} \cap M$ and $n \in \omega$ there is an $S' <^0 X$ such that the collection $\{(S')_t : (S')_t \in M$ and $(S')_t \models X \cap A_n \neq \varnothing\}$ is predense below S'.

Fact 3. If $S \in \mathbf{L} \cap M$ and $n \in \omega$ there is an $S' <^0 S$ such that the collection $\{(S')_t | (S')_t$ and $(S')_t \models A_n - X \neq \varnothing\}$ is predense below S'.

Fact 1 is, of course, a well-known property of $\mathbf{L}$ and its proof is similar to the proof of Fact 2. In the proof of Fact 2 we are just using that X is M-splitting. Since $\omega - X$ is also M-splitting, Fact 3 follows from Fact 2. Now let us prove Fact 2.

Let $B_n = \{k \in \omega | (\exists S' <^0 S) S' \models k \in A_n\}$. Let us first suppose that B_n is infinite. In this case, $B_n \in [\omega]^\omega \cap M$, hence we have that $X \cap B_n \neq \varnothing$. It follows that we may choose $S' <^0 S$, $S' \in M$ and $k \in X \cap B_N$ so that $S' \models k \in A_n$—which certainly suffices.

Now let us suppose that $\max(B_n) < m$. Let I be the set of minimal elements of $\{t \in S | (\exists m_t \geq m)(\exists S'_t <^0 X_t) S'_t \models m_t \in A_n\}$. Since $1 \models (\exists k \geq m))\ k \in A_n$ and the members of I are minimal, $\{S_t | t \in I\}$ is a maximal-below-S antichain of $\mathbf{L}$. For each $t \in I$, fix a minimal m_t and an $S'_t \in M$ as in the description of I; hence $\{m_t | t \in I\} \in M$. We shall show that $S' = \bigcup\{S'_t | t \in I$ and $m_t \in X\}$ works. That is, we prove that $S' \in \mathbf{L}$, $S' <^0 S$ and simply note that $\{S'_t | t \in I$ and $m_t \in X\}$ is a maximal-below-S' antichain. Therefore it suffices to show

that if $s \in S'$ is such that root $(S) \geq s$, then s has infinitely many immediate successors in S'. First suppose that there is a $t \in I$ with $t \leq s$. Since members of I are minimal, it follows that $(S')_s = S' \cap S_s$—hence $t \in S'$ and $m_t \in X$. Therefore $S'_t \subseteq S'$ and s has infinitely many immediate successors in S'. If there is no such $t \in I$, then S_s has no $<^0$-extension which decides a value of A_n above m. Suppose now that $k \in \omega$ is such that $\max(\{i | s^\wedge i \in S'\}) < k$. Let S'' be the $<^0$-extension of S_s obtained by removing $\{t | (\exists i < k) s^\wedge i \leq t\}$. We claim that $\{m_t | t \in I \cap S''\}$ is infinite. Indeed, since $\{S_t | t \in I \cap S''\}$ is predense below S'', $S'' \models A_n \cap \{m_t | t \in I \cap S''\} \neq \varnothing$. If the set was finite then, by $(*)$, S'' (hence S_s) would have $<^0$-extension picking one of the values. But now $X \cap \{m_t | t \in I \cap S''\}$ is nonempty, hence any $t \in I \cap S''$ with $m_t \in X$ is an extension of s in S'—a contridiction to the choice of k.

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