PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 3, March 1990

LOCALLY BOUNDED NONCONTINUOUS LINEAR FORMS ON STRONG DUALS OF NONDISTINGUISHED KÖTHE ECHELON SPACES

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this note it is proved that if $\lambda_1(A)$ is any nondistinguished Köthe echelon space of order one and $K_\infty \simeq (\lambda_1(A))'_b$ is its strong dual, then there is even a linear form $: K_\infty \to \mathbf{C}$ which is locally bounded (i.e. bounded on the bounded sets) but not continuous. It is also shown that every nondistinguished Köthe echelon space contains a sectional subspace with a particular structure.

INTRODUCTION

Following Dieudonné and Schwartz, a locally convex space E is *distinguished* if its strong dual is barrelled. Grothendieck [5] proved that a metrizable space E is distinguished if and only if E'_b is bornological. In the theory of Fréchet spaces, the notion of distinguishedness was introduced to avoid some "pathology": the naive idea that the strong dual E'_b of any Fréchet space E, i.e. of a countable projective limit of Banach spaces, must be a countable inductive limit of Banach spaces is false in general. Distinguished Fréchet spaces are those for which the idea really works. In connection with some recent investigations in Fréchet and (DF)-spaces (e.g. applications to weighted inductive limits [2], [3]; consequences of J. Taskinen's negative solution of Grothendieck's "problème des topologies" for π-tensor products of Fréchet spaces), the class of the distinguished Fréchet spaces and, in particular, the class of the distinguished Köthe echelon spaces, have again become quite important.

The first example of a nondistinguished Köthe echelon space was given by Grothendieck and Köthe [5]. It is a Köthe echelon space of order one $\lambda_1(A)$ with the Köthe matrix $A = (a_n)_{n\in\mathbf{N}}$ on the index set $I = \mathbf{N}\times\mathbf{N}$ given by $a_n(i,j) := j$ if $i \le n$, $a_n(i,j) := 1$ if $i \ge n+1$. In this case Grothendieck constructed a linear form on $\lambda_1(A)'_b$ which is locally bounded but not continuous (see [5], p. 88). Consequently the topological duals of $(\lambda_1(A))'_b$ and of its

Received by the editors May 25, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46A45; Secondary 46A05, 46A07, 46A09.
Key words and phrases. Köthe echelon spaces, Köthe matrices, distinguished spaces, locally bounded linear forms, continuous linear forms.

associated bornological space do not coincide. An example of Komura (see e.g. [6, p. 292]) shows that there are nondistinguished Fréchet spaces F such that every locally bounded linear form on F_b' is continuous.

The distinguished Köthe echelon spaces $\lambda_1(A)$ have been characterized in terms of the Köthe matrix A in [2] (see also [7]). We use here this characterization and an adaption of the original argument of Grothendieck based on [3, Appendix p. 194] to show in Theorem 2 that all nondistinguished Köthe echelon spaces $\lambda_1(A)$ have the same pathology as the Köthe–Grothendieck example. We also show in Proposition 3 that every nondistinguished Köthe echelon space contains a sectional subspace with a structure similar to the Köthe–Grothendieck example (cf. [7, 4]).

For the notations for Köthe echelon spaces, we refer to [4]. Nevertheless, we recall that we associate with a Köthe matrix $A = (a_n)_{n\in\mathbf{N}}$ on an index (not necessarily countable) set I the decreasing sequence $\mathscr{V} = (v_n)_{n\in\mathbf{N}}$ of strictly positive weights on I defined by $v_n(i) := a_n^{-1}(i)$ $(n \in \mathbf{N}, i \in I)$. The maximal Nachbin family associated to $\mathscr{V}$ is

$$\overline{V} := \left\{\overline{v}: I \to [0, +\infty[, \sup_{i\in I} \frac{\overline{v}(i)}{v_n(i)} < \infty \quad \text{for all } n \in \mathbf{N}\right\}.$$

By [4, 2.7], the space $(\lambda_1(A))_b'$ coincides algebraically and topologically with

$$K_\infty := \left\{u \in \mathbf{C}^I : p_v(u) := \sup_{i\in I} \overline{v}(i)|u(i)| < \infty \quad \text{for all } \overline{v} \in \overline{V}\right\}$$

endowed with the topology defined by the seminorms $\{p_{\overline{v}}, \overline{v} \in \overline{V}\}$. The bornological space associated to $(\lambda_1(A))_b'$ is $k_\infty := \operatorname{ind} l_\infty(v_n)$.

We first establish the converse of a result of Bierstedt and Meise [3].

Lemma 1. *Let A be a Köthe matrix on an index set I. The following conditions are equivalent*:

(a) *$\lambda_1(A)$ is not distinguished*

(b) *there are $n \in \mathbf{N}$ and a decreasing sequence $(J_k)_{k\in\mathbf{N}}$ of subsets of I such that for all $k \geq n$*

(i) $\inf_{i\in J_k} \dfrac{a_n(i)}{a_k(i)} > 0$ *while* (ii) *there is $l(k) > k$ with* $\inf_{i\in J_k} \dfrac{a_n(i)}{a_{l(k)}(i)} = 0$.

Proof. (b) $\Rightarrow$ (a) is the content of the first appendix in [3, p. 194].

(a) $\Rightarrow$ (b). According to Theorem 2.6. in [2] and to the equivalence of conditions (D) and (H) in [1], there are a sequence $\lambda_j > 0$ $(j \in \mathbf{N})$ and $n \in \mathbf{N}$ such that for all $\overline{v} \in \overline{V}$ and $m \in \mathbf{N}$ there is $i \in I$ with

$$(*) \qquad \inf(\lambda_1 v_1(i), \ldots, \lambda_m v_m(i)) \geq v_n(i) \text{ and } \overline{v}(i) < v_n(i).$$

For every $k \in \mathbf{N}$ we set $J_k := \{i \in I: \inf(\lambda_1 v_1(i), \ldots, \lambda_k v_k(i)) \geq v_n(i)\}$. Certainly $(J_k)_{k\in\mathbf{N}}$ is a decreasing sequence of subsets of I and as we have

$$\inf_{i\in J_k} \frac{v_k(i)}{v_n(i)} \geq \frac{1}{\lambda_k} \qquad \text{for all } k \in \mathbf{N}$$

condition (i) is satisfied.

Suppose there is $k \in \mathbf{N}$ such that (ii) does not hold. Then for all $l > k$ we have

$$\varepsilon_l := \inf_{i \in J_k} \frac{v_l(i)}{v_n(i)} > 0 .$$

We define

$$\alpha_j := \lambda_j \quad (j = 1, \dots, k); \qquad \alpha_j := \frac{1}{\varepsilon_j} \quad (j \geq k+1) \quad \text{and } \overline{v} := \inf_{j \in \mathbf{N}} \alpha_j v_j .$$

It is easy to see that $\overline{v} \geq v_n$ on J_k. Now, by $(*)$ there is $i \in I$ such that

$$\inf(\lambda_1 v_1(i), \dots, \lambda_k v_k(i)) \geq v_n(i) \qquad \text{and} \qquad \overline{v}(i) < v_n(i) .$$

This is a contradiction because the first inequality implies $i \in J_k$ and $\overline{v} \geq v_n$ on J_k. □

Theorem 2. *Let $\lambda_1(A)$ be a nondistinguished Köthe echelon space. Then there is a locally bounded noncontinuous linear form on $(\lambda_1(A))'_b$.*

Proof. Without loss of generality we may assume that condition (b) in Lemma 1 is satisfied with $n = 1$, $l(k) = k+1$ for all $k \in \mathbf{N}$ (this amounts to omit certain $a_n's$ if necessary), and we may also suppose that a_1 is identically 1 (by dividing by a_1). Then our hypothesis reads as follows: there is a decreasing sequence $(J_k)_{k \in \mathbf{N}}$ of subsets of I such that for all $k \in \mathbf{N}$

$$\text{(i)} \ \ \varepsilon_k := \inf_{i \in J_k} v_k(i) > 0 ;$$

$$\text{(ii)} \ \ \inf_{i \in J_k} v_{k+1}(i) = 0 .$$

For every $l \in \mathbf{N}$ and every sequence $(s_k)_{k \in \mathbf{N}}$ of strictly positive numbers, we set

$$M(l, (s_k)_{k \in \mathbf{N}}) := \bigcup_{k \geq l} \left\{ i \in J_k : v_{k+1}(i) < \frac{1}{s_k} \right\} .$$

We denote by e_i the ith unit vector in $\lambda_1(A)$. It is easy to see that the family of subsets of $\lambda_1(A)$

$$\{e_i \in \lambda_1(A) : i \in M(l, (s_k)_{k \in \mathbf{N}})\}$$

is a basis for a filter $\mathscr{M}$ on $\lambda_1(A)$. From now on, the space $\lambda_1(A)$ will be considered as a subspace of the algebraic dual K_∞^* of K_∞. Let $\mathscr{U}$ be an ultrafilter in $\lambda_1(A)$ containing $\mathscr{M}$.

For every $u \in K_\infty$, the family of sets

$$M(l, (s_k)_{k \in \mathbf{N}}, u) := \{\langle u, e_i \rangle = u(i) : i \in M(l, (s_k)_{k \in \mathbf{N}})\}$$

is a basis for an ultrafilter $\mathscr{U}(u)$ on $\mathbf{C}$.

Given $u \in K_\infty$, there are $M > 0$ and $n \in \mathbf{N}$ such that

$$\sup_{i \in I} v_n(i) |u(i)| \leq M .$$

If $l > n$ and $s_k > 0$ $(k \in \mathbf{N})$, we have

$$|\langle u, e_i \rangle| = |u(i)| \leq M v_n^{-1}(i) \leq M \varepsilon_n^{-1}$$

for all $i \in M(l, (s_k)_{k \in \mathbf{N}})$, since $i \in J_k \subset J_n$ $(k \geq 1)$. Therefore $M(l, (s_k)_{k \in \mathbf{N}}, u)$ is bounded in $\mathbf{C}$. Then $\mathscr{U}(u)$ converges in $\mathbf{C}$ to an element $f(u)$, from where it follows that the ultrafilter $\mathscr{U}$ converges to f in $(K_\infty^*, \sigma(K_\infty^*, K_\infty))$.

If $u \in K_\infty$ satisfies $\sup_{i \in I} v_n(i)|u(i)| \leq \varepsilon_n$, then for every $l \geq n$ and every $i \in M(l, (s_k)_{k \in \mathbf{N}})$ we have $i \in J_n$ thus also $|\langle u, e_i \rangle| = |u(i)| \leq \varepsilon_n v_n^{-1}(i) \leq 1$. This implies

$$|f(u)| \leq 1 \qquad \text{for every } u \in \Gamma\left(\bigcup_{n \in \mathbf{N}} \left\{u \in K_\infty : \sup_{i \in I} v_n(i)|u(i)| \leq \varepsilon_n\right\}\right).$$

Consequently f is locally bounded and f belongs to k'_∞.

We prove that f is not continuous on K_∞. Indeed: if it were, we could find $\overline{v} \in \overline{V}$ such that

$$|f(u)| \leq \sup_{i \in I} \overline{v}(i)|u(i)| \qquad \text{for every } u \in K_\infty.$$

We may assume that $\overline{v} = \inf_{k \in \mathbf{N}} \alpha_k v_k$ for certain $\alpha_k > 0$ $(k \in \mathbf{N})$.

We first observe that if i belongs to the set $M(1, (2\alpha_{k+1})_{k \in \mathbf{N}})$ then $\overline{v}(i) \leq 2^{-1}$.

Now we define $u(i) := 2$ if $i \in M(1, (2\alpha_{k+1})_{k \in \mathbf{N}})$ and $u(i) := 0$ otherwise. Certainly the sequence $u := (u(i))_{i \in I}$ belongs to $l_\infty(v_1) \subset k_\infty = K_\infty$. Moreover we have

$$\sup_{i \in I} \overline{v}(i)|u(i)| = \sup_{i \in M(1, (2\alpha_{k+1})_{k \in \mathbf{N}})} \overline{v}(i)|u(i)| \leq 1.$$

Therefore $|f(u)| \leq 1$. But $f(u)$ belongs to the closure in $\mathbf{C}$ of the set $\{\langle u, e_i \rangle : i \in M(1, (2\alpha_{k+1})_{k \in \mathbf{N}})\}$ and for every $i \in M(1, (2\alpha_{k+1})_{k \in \mathbf{N}})$ we have $\langle u, e_i \rangle = 2$. Consequently $f(u) = 2$, which contradicts $|f(u)| \leq 1$. □

Our next proposition, which is a consequence of Lemma 1, was observed by C. Fernandez.

Proposition 3. *Let A be a Köthe matrix on an index set I. The following conditions are equivalent*:

(a) $\lambda_1(A) = \lambda_1(I, A)$ *is not distinguished.*

(b) *there is a sectional subspace of $\lambda_1(A)$ isomorphic to a Köthe echelon space $\lambda_1(\mathbf{N} \times \mathbf{N}, B) = \lambda_1(B)$ where the matrix $B = (b_n)_{n \in \mathbf{N}}$ satisfies*

$$(1)\ b_n(k, j) = b_1(k, j) \qquad \textit{for } n \leq k \textit{ and}$$

$$(2)\ \lim_{j \to \infty} \frac{b_n(n, j)}{b_{n+1}(n, j)} = 0 \qquad (\text{cf. } [7, 4]).$$

Proof. (b) implies (a) as a consequence of [7, 4] where it is proved that these two conditions on B imply that $\lambda_1(B)$ is not distinguished.

For the converse we apply Lemma 1 and we proceed as in the proof of Theorem 2 to find a decreasing sequence $(J_k)_{k\in\mathbf{N}}$ of subsets of I such that

$$(1)\ a_1(i) = 1 \text{ for all } i,$$
$$(2)\ \varepsilon_k := \inf\left\{a_k^{-1}(i): i \in J_k\right\} > 0,$$
$$(3)\ \inf\left\{a_{k+1}^{-1}(i): i \in J_k\right\} = 0 \qquad \text{for all } k \in \mathbf{N}.$$

By (2) and (3), given $k \in \mathbf{N}$ we can find a sequence $(i(k,j))_{j\in\mathbf{N}}$ of different elements of $J_k \backslash J_{k+1}$ such that $\lim_{j\to\infty} a_{k+1}^{-1}(i(k,j)) = 0$. Now we define $J := \{i(k,j): k,j \in \mathbf{N}\}$ and the Köthe matrix $B = (b_n)_{n\in N}$ by $b_n(k,j) := a_n(i(k,j))$ if $k < n$ and by $b_n(k,j) := 1$ if $k \geq n$. By construction, $\lambda_1(J, A_{|J})$ is a sectional subspace of $\lambda_1(I, A)$ and it is easy to see that B satisfies conditions (1) and (2). Moreover, the canonical map

$$\Psi: \lambda_1(J, A_{|J}) \to \lambda_1(\mathbf{N}\times\mathbf{N}, B) \quad y \mapsto (y(i(k,j)))_{j,k\in\mathbf{N}}$$

is a topological isomorphism. Indeed, as we have $b_n(k,j) \leq a_n(i(k,j))$ for every $k, j \in \mathbf{N}$, this map is well defined and continuous. Using condition (2) above and the definition of B, it is also direct to prove that Ψ^{-1} is well defined and continuous. Thus the proof is complete. □

Remark 4. Our theorem provides a class of examples which are relevant from another point of view. In fact, if $\lambda_1(A)$ is any nondistinguished Köthe echelon space, then $(\lambda_1(A)'', \sigma(\lambda_1(A)'', \lambda_1(A)'))$ is quasi-Suslin but not K-Suslin.

Indeed, this space is quasi-Suslin by [6, Chap. 1, 4.3.(23)] and it is not K-Suslin by [6, Chap. 1, 4.3.(24)] if we observe that according to Theorem 2, the space $(\lambda_1(A)', \mu(\lambda_1(A)', \lambda_1(A)''))$ is not bornological, hence not barrelled. □

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