PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 3, March 1990

10_{101} HAS NO PERIOD 7: A CRITERION FOR PERIODIC LINKS

PAWEL TRACZYK

(Communicated by Haynes R. Miller)

ABSTRACT. We prove that the prime knot 10_{101} has no period 7 by analyzing its Jones polynomial.

The purpose of this paper is to introduce a convenient criterion for periodicity of links in S^3. Our tool will be the Kauffman bracket polynomial $\langle\,\rangle$ taking values in $Z[A^{-1}, A]$ and the normalized version f which is an ambient isotopy invariant with values in $Z[A^{-2}, A^2]$ (see [K]). Let us define a reduction homomorphism from $Z[A^{-1}, A]$ into the group ring over Z_p of the multiplicative cyclic group C_{p^n} by reducing the coefficients modulo p and putting $A \to \gamma$, where $\langle\gamma\rangle = C_{p^n}$. We will denote by $f_{p,n}$ the image of f under this homomorphism. Then we have the following.

Theorem. *If L has period p^n, $n \geq 1$ (p a prime), then $f_{p,n}(L)$ is a symmetric element in $Z_p[C_{p^n}]$, that is the coefficient of $\alpha \in C_{p^n}$ is the same as that of α^{-1}.*

Corollary. *10_{101} has no period 7: The f polynomial of 10_{101} is $A^{-48} - A^{-44} + 7A^{-40} - 11A^{-36} + 13A^{-32} - 14A^{-28} + 14A^{-24} - 10A^{-20} + 7A^{-16} - 3A^{-12} + A^{-8}$, so the reduced version is $3A^{-2} + 4A^{-1} + 5A + 4A^2 + 6A^3$ which is not symmetric.*

Murasugi [M] has recently shown, that 10_{105}—one of the other undecided cases in Burde-Zieschang table (see [B-Z])—has no period 7. Our criterion also works in this case, the reduced polynomial being equal $A^{-2} + 5A^{-1} + 4 + 6A + 3A^2 + 3A^3$.

Proof of the theorem. We will work with a periodic diagram D of the considered periodic link. That is D is invariant under the rotation ρ of the projection plane around the center 0 disjoint from D. This makes it possible to restrict our attention to the bracket polynomial rather than the f polynomial, because by definition $f(L) = (-A)^{-3w(L)}\langle L\rangle$, and for a periodic diagram of an oriented link L the twist number $w(L)$ is obviously divisible by p^n, whence

Received by the editors March 13, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57M25; Secondary 57S17.

$(-A)^{-3w(L)} = \pm 1$ and $f_{p,n}(L) = \pm\langle L\rangle_{p,n}$. Thus we only need to prove that the reduced bracket polynomial of a periodic diagram is symmetric.

In computing the bracket polynomial of D we consider the states of D, which are obtained by splitting every crossing of the diagram in one of the two possible ways (a positive split $\times \to \asymp$, the NW-SE segment being an overpass, or a negative split $\times \to)($, the NW–SE segment being an overpass). Each of the states has its contribution in the bracket expansion: every split scores A or A^{-1}, and the product of these coefficients multiplied by $(-A^{-2} - A^2)^{|S|-1}$ is the contribution of S in the bracket polynomial of D ($|S|$ denotes the number of components). Thus the contribution of S is a summand of the form $A^k(-A^{-2} - A^2)^{|S|-1}$. The group $\langle\rho\rangle$ acts on the set of states of D. Obviously the contributions of two states belonging to one orbit are equal. Thus the total contribution of a nontrivial orbit to the reduced bracket is trivial (because there are p^k equivalent states in such an orbit, $k \geq 1$).

Let us consider a periodic state S. If this state is obtained by a positive split at a crossing c, then the invariance of S implies that all the images of c under the action of $\langle\rho\rangle$ are also positively split. Thus the splits at these crossings contribute A^{p^n} to the bracket of S and this is equal 1 in the reduced situation. It follows that the contribution of S to the reduced bracket is simply $\langle S\rangle = (-A^{-2} - A^2)^{|S|-1}$, and this is obviously symmetric. This completes the proof.

References

[B-Z] G. Burde and H. Zieschang, *Knots*, De Gruyter, 1985.

[K] L. H. Kauffman, *State models for knot polynomials*, Topology, **26** (1987), 395–407.

[M] K. Murasugi, *Jones polynomials of periodic links*, Pacific J. Math. **131** (2) (1988), 319–329.

Institute of Mathematics, Warsaw University, 00-901 Warsaw, Poland, PKiN IXp