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COPIES OF l_∞ IN $L^p(\mu; X)$

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ABSTRACT. Let X be a Banach space and let (Ω, Σ, μ) be a measure space. For $1 \leq p < +\infty$ we denote by $L^p(\mu; X)$ the Banach space of all X-valued Bochner p-integrable functions on Ω. In this note we show that $L^p(\mu; X)$ contains an isomorphic copy of l_∞ if and only if X does.

Let X be a Banach space and let (Ω, Σ, μ) be a measure space. For $1 \leq p < +\infty$ we denote by $L^p(\mu; X)$ the Banach space of all X-valued Bochner p-integrable functions on Ω. The aim of this note is to prove the following:

Theorem. *Let $1 \leq p < +\infty$, then $L^p(\mu; X)$ contains an isomorphic copy of l_∞ if and only if X does.*

This result completes in some way the results given in [5] and [7] where conditions for $L^p(\mu; X)$ to contain l_1 or c_o are given. Our proof is strongly inspired by [2], where several ideas from [4] are used. In fact, the three lemmas we need in our proof may be found in [2] (although only the last one appears there for the first time.)

If A is a set, we denote by $[A]$ the set of infinite subsets of A. Let $\mathbf{N}$ be the set of natural numbers; for $M \in [\mathbf{N}]$, $l_\infty(M)$ is defined to be the subspace of l_∞ of all sequences $(\zeta_n) \in l_\infty$ with $\zeta_k = 0$ for $k \notin M$. We denote by $\{e_n\}$ the unit vector sequence of l_∞.

Lemma 1 (Proposition 1.2 and Remark 1 of [8]). *If $T\colon l_\infty \to X$ is an operator such that $\|T(e_n)\|_\infty \nrightarrow 0$, then there is $M \in [\mathbf{N}]$ such that $T|_{l_\infty(M)}$ is an isomorphism.*

Lemma 2 (Corollary 1.4 of [8]). *If X contains no copy of l_∞, then every operator $T\colon l_\infty \to X$ is weakly compact.*

Lemma 3 ([2]). *Let $\{T_k\}$ be a sequence of weakly compact operators from l_∞ into X; then there exists $M \in [\mathbf{N}]$ such that $T_k((\zeta_n)) = \sum_{n=1}^{\infty} \zeta_n T_k(e_n)$ for all $(\zeta_n) \in l_\infty(M)$ and all $k \in \mathbf{N}$.*

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Proof of the theorem. The *if* part of the result is of course trivial. For the converse we shall prove first the following:

(*) If $L^p(\mu; X)$ contains a copy of l_∞, then there is a σ-finite measure space $(\Omega_1, \Sigma_1, \mu_1)$ such that

(i) Σ_1 is generated by a sequence of measurable sets of finite measure (and therefore, $L^p(\mu_1)$ is separable), and

(ii) $L^p(\mu_1; X)$ contains a copy of l_∞.

Let us suppose that $L^p(\mu; X)$ contains a copy of l_∞ and let J be an isomorphic embedding from l_∞ into $L^p(\mu; X)$. By III.8.5. of [3] we know that there is a σ-finite measure space $(\Omega_1, \Sigma_1, \mu_1)$, such that

(a) $\Omega_1 \in \Sigma, \Sigma_1 \subset \Sigma(\Omega_1) = \{A \in \Sigma : A \subset \Omega_1\}$, and $\mu_1 = \mu|_{\Sigma_1}$,

(b) $\{J(e_n)\} \subset L^p(\mu_1; X)$, and

(c) Σ_1 is generated by a sequence of measurable sets of finite measure (and therefore, $L^p(\mu_1)$ is separable).

Let R be the restriction operator from $L^p(\mu; X) = L^p(\Omega, \Sigma, \mu, X)$ into $L^p(\Omega_1, \Sigma(\Omega_1), \mu|_{\Sigma(\Omega_1)}; X)$ and let E be the conditional expectation operator from $L^p(\Omega_1, \Sigma(\Omega_1), \mu|_{\Sigma(\Omega_1)}; X)$ into $L^p(\mu_1; X) = L^p(\Omega_1, \Sigma_1, \mu_1; X)$ (see f.i. V.1. of [1]). If we take $J_o = E \circ R \circ J$ it is clear that

$$\|J_o(e_n)\|_p = \|J(e_n)\|_p \nrightarrow 0.$$

Then, by Lemma 1, $L^p(\mu_1; X)$ contains a copy of l_∞. This proves (*).

Let us suppose now that $L^p(\mu; X)$ contains a copy of l_∞ and that X does not. We can assume, by (*), that μ is a σ-finite measure on a σ-field Σ generated by a sequence $\{A_k\}$ of sets of finite measure.

Let J be an isomorphic embedding from l_∞ into $L^p(\mu; X)$, and let us define for each $k \in \mathbf{N}$

$$J_k : l_\infty \to X$$
$$(\zeta_n) \to \int_{A_k} J((\zeta_n))\, d\mu$$

By Lemma 2, $\{J_k\}$ is a sequence of weakly compact operators, and, by Lemma 3, there is $M \in [\mathbf{N}]$, such that

$$J_k((\zeta_n)) = \sum_{n=1}^{\infty} \zeta_n J_k(e_n) \text{ for all } (\zeta_n) \in l_\infty(M) \text{ and all } k \in \mathbf{N}.$$

Let $J_o = J|_{l_\infty(M)}$, and let X_o be a separable subspace of X, such that

$$J(e_n)(t) \in X_o \qquad \mu\text{-almost everywhere, for every } n \in M.$$

Let $(\zeta_n) \in l_\infty(M)$, and let $k \in \mathbf{N}$; we have

$$\int_{A_k} J_o((\zeta_n))\, d\mu = \int_{A_k} J((\zeta_n))\, d\mu = J_k((\zeta_n)) = \sum_{n=1}^{\infty} \zeta_n J_k(e_n) = \sum_{n=1}^{\infty} \zeta_n \int_{A_k} J(e_n)\, d\mu .$$

Therefore,

$$\int_{A_k} J_o((\zeta_n))\, d\mu \in X_o, \qquad \text{for all } k \in \mathbf{N}.$$

Since Σ is generated by $\{A_k\}$, standard arguments allow us to conclude that

$$\int_A J_o((\zeta_n))\, d\mu \in X_o, \qquad \text{if } A \in \Sigma \text{ and } \mu(A) < +\infty .$$

Then we have (see f.i. X, §5, Theorem 5 of [6])

$$J_o((\zeta_n))(t) \in X_o \qquad \mu\text{-almost everywhere},$$

and so,

$$J_o((\zeta_n)) \in L^p(\mu; X_o), \qquad \text{for every } (\zeta_n) \in l_\infty(M) .$$

But this implies that J_o is an isomorphic embedding of $l_\infty(M) \approx l_\infty$ into the separable space $L^p(\mu; X_o)$. This contradiction finishes the proof.

References

1. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys No. 15, Amer. Math. Soc., Providence, RI, 1977.
2. L. Drewnowski, *Copies of l_∞ in the operator space $K_{w^*}(X^*, Y)$*, (to appear).
3. N. Dunford and J. T. Schwartz, *Linear operators*, vol. I. New York, Interscience, 1955.
4. N. J. Kalton, *Spaces of compact operators*, Math. Ann. **208** (1974), 267–278.
5. S. Kwapien, *Sur les espaces de Banach contenant c_0*, Studia Math. **52** (1974), 187–188.
6. S. Lang, *Analysis* II, Addison-Wesley, Reading, MA, 1969.
7. G. Pisier, *Une propriété de stabilité de la classe des espaces ne contenant pas l_1*, C. R. Acad. Sci. Paris Sér. A **286** (1978), 747–749.
8. H. P. Rosenthal, *On relatively disjoint families of measures with some applications to Banach space theory*, Studia Math. **37** (1970), 13–16.

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