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CONVOLUTION IN THE HARMONIC HARDY CLASS h^p WITH $0 < p < 1$

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ABSTRACT. It is proved that if $u \in h^p$, $0 < p < 1$, and $v \in h^q$, $q \geq p$, then

$$M_q(u * v, r) = 0\left((1-r)^{1-1/p}\right), \quad r \to 1-,$$

where $u * v$ stands for the convolution of u and v.

1. INTRODUCTION

Let $h(\Delta)$ denote the class of all (complex-valued) harmonic functions in the unit disc Δ of the complex plane $\mathbf{C}$. The harmonic Hardy class h^p, $0 < p < \infty$, is the subclass of $h(\Delta)$ consisting of those functions u for which

$$\|u\|_p := \sup_{0<r<1} M_p(u, r) < \infty.$$

Here $M_p(u, r)$ is the pth mean value of u:

$$M_p^p(u, r) = \int_T |u(rw)|^p \, d\delta(w),$$

where T is the unit circle and $d\delta$ is the normalized Lebesgue measure on T.

The Hardy space H^p is the closed subspace of h^p consisting of functions holomorphic in Δ.

For $p \geq 1$, the structure of h^p spaces is well known (see [1], Chapters 2, 3, and 4) and is similar to the structure of H^p and L^p spaces. The fundamental results concerning the case $p < 1$ were obtained by Hardy and Littlewood [2]; for further information and references we refer to [5].

Each $u \in h(\Delta)$ has a unique series expansion

$$u(rw) = \sum_{k=-\infty}^{\infty} \hat{u}(k) r^{|k|} w^k, \qquad w \in T, \qquad 0 < r < 1,$$

which converges uniformly and absolutely on compact subsets of Δ. The convolution of u and $v (\in h(\Delta))$ is defined by

$$(u * v)(rw) = \sum_{k=-\infty}^{\infty} \hat{u}(k)\hat{v}(k) r^{|k|} w^k, \qquad w \in T, \qquad 0 < r < 1,$$

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which can be written as

$$(1.1) \qquad (u * v)(r^2 w) = \int_T u(rw\zeta)v(r\overline{\zeta})\, d\delta(\zeta), \qquad w \in T, \qquad 0 < r < 1.$$

This identity along with Minkowski's inequality in continuous form is used to prove that

$$M_p\left(u * v, r^2\right) \leq M_p(u, r)M_1(v, r), \qquad p \geq 1.$$

In particular, if $u \in h^p$, $p \geq 1$, and $v \in h^1$, then $u * v \in h^p$. This fact is extended by the following theorem.

Theorem 1.1. *If $0 < p < 1$ and $q \geq p$, then there is a constant $C = C_{p,q} < \infty$ such that*

$$(1.2) \qquad M_q(u * v, r) \leq C(1-r)^{1-1/p} \|u\|_p \|v\|_q, \qquad 0 < r < 1,$$

for all $u \in h^p$, $v \in h^q$.

The analytic case of this theorem is considered in [3], and the proof is based on the inequality

$$(1.3) \qquad M_1(F, r) \leq C_p(1-r)^{1-1/p} M_p\left(F, \frac{1+r}{2}\right) \qquad (p < 1),$$

which is valid for analytic functions.

In §1 we use a certain maximal function to obtain a more general version of (1.3). However, the heart of our proof of Theorem 1.1 is the following fundamental result of Hardy and Littlewood [2].

Theorem HL. *Let U be a harmonic function in a disc D centered at z_0. Then, for each $p > 0$,*

$$|U(z_0)|^p \leq C_p m(D)^{-1} \int_D |U|^p\, dm,$$

where C_p is a constant depending only on p, and dm stands for the Lebesgue measure in the plane.

A short proof of Theorem HL is given in [4].

Theorem 1.1 is a consequence of a stronger result that will be proved in §3. Also, §3 contains a generalization of Theorem 1.1 to a class of weighted h^p spaces.

2. A maximal function

For a complex-valued function F continuous in the unit disc Δ we define the function F^+ by

$$(2.1) \qquad F^+(r\zeta) = \sup\{|F(rz)| : |z - \zeta| \leq 1 - r,\ z \in T\},\ \zeta \in T,\ r < 1.$$

(Note that the supremum is taken over an arc.) This maximal function differs from (and is much simpler than) the radial maximal function of Hardy and Littlewood.

Lemma 2.1. *Let* $0 < p < 1$, *and let* F *be a function continuous in* Δ. *Then*

$$M_1(F, r) \leq C(1-r)^{1-1/p} M_p(F^+, r), \qquad 0 < r < 1, \tag{2.2}$$

where C *depends only on* p.

Proof. First we prove that

$$M_\infty(F, r) \leq C(1-r)^{-1/p} M_p\left(F^+, r\right), \qquad 0 < r < 1, \tag{2.3}$$

where

$$M_\infty(F, r) = \sup\{|F(rw)| : w \in T\}.$$

For a fixed r, $0 < r < 1$, let $T_\zeta = \{w \in T : |w - \zeta| \leq 1 - r\}$. It follows from (2.1) that if $w \in T_\zeta$, then

$$F^+(rw) = \sup\left\{|F(rz)| : w \in T_z\right\} \geq |F(r\zeta)|.$$

Hence

$$\begin{aligned} M_p^p(F^+, r) &= \int_T F^+(rw)^p \, d\delta(w) \\ &\geq \int_{T_\zeta} |F(r\zeta)|^p \, d\delta(w) = |F(r\zeta)|^p \, \delta(T\zeta). \end{aligned}$$

But $\delta(T_\zeta)$ is independent of ζ and equals $\frac{2}{\pi} \arcsin(\frac{1-r}{2})$, whence

$$M_p^p\left(F^+, r\right) \geq |F(r\zeta)|^p (1-r)/\pi,$$

which proves (2.3). We see that (2.3) holds for all $p > 0$. If $p < 1$, then

$$\begin{aligned} M_1(F, r) &\leq M_\infty^{1-p}(F, r) M_p^p(F, r) \\ &\leq M_\infty^{1-p}(F, r) M_p^p(F^+, r), \end{aligned}$$

and now (2.2) follows from (2.3). □

When $p \geq 1$, there is a simple relation between the pth mean values of a harmonic function U and U^+. Namely, it is easy to see that for the Poisson kernel P, we have $P^+ \leq CP$, which implies that

$$M_p\left(U^+, r\right) \leq C M_p\left(U, \frac{1+r}{2}\right) \qquad (p \geq 1) \tag{2.4}$$

where C is an absolute constant. In the case $p < 1$ we must use some other mean values. For $0 < p < \infty$ we let

$$A_p(F, r) = \left\{(1-r)^{-2} \int_{r_0}^{r_1} M_p^p(F, \varphi) \, d\varphi\right\}^{1/p},$$

where

$$\begin{aligned} r_0 &= \max\{0, r - (1-r)/2\} \\ r_1 &= r + (1-r)/2 = (1+r)/2, \qquad 0 < r < 1. \end{aligned}$$

Note that $A_p(F, r)$ is "equivalent" to the pth mean value of F over the annulus $r_0 \leq |z| \leq r_1$.

Lemma 2.2. *If $p > 0$ and if U is harmonic in Δ, then there is a constant $C = C_p$ such that*

$$M_p\left(U^+, r\right) \leq C A_p(U, r), \qquad 0 < r < 1. \tag{2.5}$$

This implies (2.4) *because of the increasing property of $M_p(U, r)$, $p \geq 1$.*

Proof. For fixed r and $z \in T$ let $D = \{w \in \mathbf{C}: |w - rz| \leq (1-r)/2\}$. By Theorem HL, we have

$$|U(rz)|^p \leq C(1-r)^{-2} \int_D |U(w)|^p \, dm(w). \tag{2.6}$$

If $w \in D$, then

$$|1 - rw\overline{z}| \leq (3/2)(1-r).$$

From this and (2.6) we find that

$$|U(rz)|^p \leq C \int_D |U(w)|^p |1 - rw\overline{z}|^{-2} \, dm(w).$$

Since D is contained in the annulus

$$A(r) = \{w \in \mathbf{C}: r - (1-r)/2 \leq |w| \leq r + (1-r)/2\}$$

we get

$$|U(rz)|^p \leq C \int_{A(r)} |U(w)|^p |1 - rw\overline{z}|^{-2} \, dm(w), \qquad z \in T, \tag{2.7}$$

where C depends only on p. On the other hand, if $|z| = |\zeta| = 1$ and $|z - \zeta| \leq 1 - r$, then for all $w \in \Delta$, we have

$$\left|1 - rw\overline{\zeta}\right| \leq 2|1 - rw\overline{z}|.$$

From this, (2.7), and (2.1) we see that

$$U^+(r\zeta)^p \leq C \int_{A(r)} |U(w)|^p \left|1 - rw\overline{\zeta}\right|^{-2} dm(w), \qquad |\zeta| = 1.$$

Hence, by integration over the circle $|\zeta| = 1$,

$$M_p^p\left(U^+, r\right) \leq C \int_{A(r)} |U(w)|^p (1 - r|w|)^{-1} \, dm(w).$$

Integration in polar coordinates concludes the proof. □

3. Mean values of $u * v$

Theorem 1.1 is an immediate consequence of the following:

Theorem 3.1. *If $0 < p < 1$ and $q \geq p$, then there is a constant $C = C_{p,q}$, such that*

$$M_q\left(u * v, r^2\right) \leq C(1-r)^{1-1/p} A_p(u, r) A_q(v, r), \qquad 0 < r < 1, \tag{3.1}$$

for all $u, v \in h(\Delta)$.

Proof. By Lemma 2.2, it suffices to prove that

$$M_q\left(u * v, r^2\right) \leq C(1-r)^{1-1/p} M_p\left(u^+, r\right) M_q\left(v^+, r\right). \tag{3.2}$$

Let $0 < p < 1$ and $q \geq p$. For a fixed $w \in T$ let

$$F(r\zeta) = u\left(r\bar{\zeta}\right) v(rw\zeta), \qquad 0 < r < 1, \qquad \zeta \in T.$$

Then

$$(u * v)\left(r^2 w\right) = (v * u)\left(r^2 w\right) = \int_T F(r\zeta)\, d\delta(\zeta)$$

(Here u and v are harmonic, but the proof of (3.2) is independent of this hypothesis.) By Lemma 2.1,

$$\begin{aligned}\left|(u * v)\left(r^2 w\right)\right|^p &\leq M_1^p(F, r) \\ &\leq C(1-r)^{p-1} \int_T F^+(r\zeta)^p\, d\delta(\zeta).\end{aligned}$$

Since obviously $F^+(r\zeta) \leq u^+(r\bar{\zeta}) v^+(rw\zeta)$, we get

$$\left|(u * v)\left(r^2 w\right)\right|^p \leq C(1-r)^{p-1} \int_T u^+\left(r\bar{\zeta}\right)^p v^+(rw\zeta)^p\, d\delta(\zeta).$$

Hence, by Minkowski's inequality for M_s, $s = q/p \geq 1$,

$$\begin{aligned}M_q^p\left(u * v, r^2\right) &= M_S\left(|u * v|^p, r^2\right) \\ &\leq C(1-r)^{p-1} \int_T \left[\int_T u^+\left(r\bar{\zeta}\right)^{ps} v^+(rw\zeta)^{ps}\, d\delta(w)\right]^{1/s} d\delta(\zeta) \\ &= C(1-r)^{p-1} \int_T u^+\left(r\bar{\zeta}\right)^p d\delta(\zeta) \left[\int_T v^+(rw\zeta)^q\, d\delta(w)\right]^{p/q} \\ &= C(1-r)^{p-1} M_p^p\left(u^+, r\right) M_p^p\left(v^+, r\right),\end{aligned}$$

which was to be proved. □

The following generalization of Theorem 1.1 follows directly from Theorem 3.1. Here

$$h^p(\alpha) = \left\{u \in h(\Delta) : M_p(u, r) = 0\left((1-r)^{-\alpha}\right), r \to 1\right\}$$

and

$$h_0^p(\alpha) = \left\{u \in h(\Delta) : M_p(u, r) = 0\left((1-r)^{-\alpha}\right), r \to 1\right\},$$

where α is a real number and $0 < p \leq \infty$. Thus $h^p(0) = h^p$, while $h_0^p(0)$ coincides with the space h_0^p which was considered by Shapiro [5].

Theorem 3.2. *Let* $0 < p < 1$ *and* $q \geq p$. *Then*

(i) *If* $u \in h^p(\alpha)$ *and* $v \in h^q(\beta)$, *then* $u * v \in h^q(\alpha + \beta + 1/p - 1)$;

(ii) *If* $u \in h_0^p(\alpha)$ (*resp.* $u \in h^p(\alpha)$) *and* $v \in h^q(\beta)$ (*resp.* $v \in h_0^q(\beta)$), *then* $u * v \in h_0^q(\alpha + \beta + 1/p - 1)$.

In contrast to the case of analytic functions, the spaces $h^p(\alpha)$ $(0 < p < 1)$ need not be trivial for $\alpha < 0$. For example, we have the following interesting fact.

Corollary 3.1. *If* $1/2 < p < 1$, *then the space* $h^p(1 - 1/p)$ *is an infinite-dimensional algebra* (*relative to the operation* $*$) *with the Poisson kernel as unit.*

Proof. That $h^p(1 - 1/p)$ is an algebra follows from Theorem 3.2(i). The rest follows from the fact that all the functions

$$P_w(z) = \frac{1 - |z|^2}{|1 - wz|^2}, \qquad w \in T, \qquad z \in \Delta,$$

(the rotates of the Poisson kernel $P = P_w$, $w = 1$) belong to $h^p(1 - 1/p)$. See [5] and [4]. □

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