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AN ASYMPTOTIC BOUND FOR THE ITERATES OF CERTAIN REAL FUNCTIONS NEAR A CONTRACTIVE FIXED POINT

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ABSTRACT. If $x = 0$ is a contractive fixed point for the function F, then under certain conditions, the iterates $F_k(a)$ are asymptotically equal to the numbers ξ_k defined by $k = \int_{\xi_k}^{a} \frac{du}{u-F(u)}$. Using somewhat different hypotheses, we give a more precise bound on $F_k(a)/\xi_k$.

Let F be a real function of the real variable x, defined and continuous for $0 \le x \le A$, with $0 < F(x) < x$ for $0 < x \le A$. The kth iterate of F will be denoted by $F_k(F_0(x) \equiv x)$. Set the error $x - F(x) = G(x)$. In the classical case where $G(x) \sim Cx^{\alpha}$, $C > 0$, $\alpha > 1$, the asymptotic formula $F_k(a) \sim ((\alpha - 1)Ck)^{1/1-\alpha}$ has been known for a long time (see [1] or [3]).

In the elementary note [2], the author and V. Drobot dealt with certain other cases and proved that when F converges to 0 "slowly" and certain other conditions are met, then the first asymptotic term is given by ξ_k where ξ_k is defined by the equation

$$(*) \qquad k = \int_{\xi_k}^{a} \frac{du}{G(u)}.$$

In addition, we gave several heuristic arguments for $(*)$. The theorem proved in [2] follows.

Theorem 0. *With the notation given previously, assume $G \in C^1$, $G(0) = G'(0) = 0$, and G is strictly increasing. Define*

$$H(x) = \int_{x}^{a} \frac{du}{G(u)}$$

so that $H(\xi_k) = k$. Assume that for each $\mu > 1$,

$$\limsup_{x \to 0} \frac{H(\mu x)}{H(x)} < 1.$$

Then $\xi_k \sim F_k(a)$.

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The main purpose of this note is to give a bound for $F_k(a)/\xi_k$. We do this by imposing a new criterion for "slow convergence to 0." Moreover, under this condition, we can dispense with the unwanted assumption that $G'(0) = 0$. In fact, the result will apply to certain functions composed of line segments whose slopes take successively the values 0 and 1. Although, in the generality of our setting, we cannot get a true second asymptotic term, our error has the correct order of magnitude in the classical case.

Theorem 1. *With the notation from above, assume F and G are increasing for $0 \le x \le A$. Let K denote the function inverse to H, and set $Q = -K'$. Assume moreover that*

$$\lim_{\theta \downarrow 1} \limsup_{x \to \infty} Q(x)/Q(\theta x) = 1.$$

Then there is a positive constant C, so that if $k \ge 1$ and $\xi_{\ell+1} \le F_k(a) < \xi_\ell$, then $\ell \ge k > \ell - C \log \ell$ and $\xi_k \sim F_k(a)$. Moreover, C can be chosen independent of a for $a \le a_0$.

Remarks. The hypothesis on Q is equivalent to the condition that $\log Q$ be slowly oscillating. The conditions imposed on F and G trivially imply that for $0 < x < y < A$, $0 < F(y) - F(x) < y - x$. Since $0 < G(x) < x$ and is increasing, K is defined for $0 \le x < \infty$ and is convex. Thus $Q = -K' = G(K)$. We denote $F_k(a)$ by η_k.

Proof of Theorem 1. First, note that $F(\xi_\ell) < \xi_{\ell+1}$, for $\xi_{\ell+1} - F(\xi_\ell) = \xi_{\ell+1} - \xi_\ell + G(\xi_\ell) = K(\ell+1) - K(\ell) - K'(\ell) > 0$ by the convexity of K. If $\eta_j < \xi_\ell$, then $\eta_{j+1} = F(\eta_j) < F(\xi_\ell) < \xi_{\ell+1}$, so that every interval $[\xi_{\ell+1}, \xi_\ell)$ contains at most one η.

Now suppose $\eta_{k+1} < \xi_{\ell+1} < \xi_\ell \le \eta_k$. We will prove inductively that for $n > 0$,

$$(**) \qquad 0 < \xi_{\ell+n} - \eta_{k+n} \le K(\ell+n) - K(\ell) - \sum_{j=0}^{n-1} K'(\ell+j)$$

with strict inequality for $n > 1$.

To start the induction we have $F(\xi_\ell) \le \eta_{k+1} < \xi_{\ell+1}$, so $0 < \xi_{\ell+1} - \eta_{k+1} \le \xi_{\ell+1} - F(\xi_\ell) = K(\ell+1) - K(\ell) - K'(\ell)$. Assuming $(**)$ for n, we have $\eta_{k+n} < \xi_{\ell+n}$ and so $\eta_{k+n+1} < F(\xi_{\ell+n}) < \xi_{\ell+n+1}$. Using the Lipschitz condition satisfied by F and the induction hypothesis, we get

$$\begin{aligned}
0 < \xi_{\ell+n+1} - \eta_{k+n+1} &= \xi_{\ell+n+1} - F(\xi_{\ell+n}) + F(\xi_{\ell+n}) - F(\eta_{k+n}) \\
&< \xi_{\ell+n+1} - \xi_{\ell+n} + G(\xi_{\ell+n}) + (\xi_{\ell+n} - \eta_{k+n}) \\
&\le K(\ell+n+1) - K(\ell+n) - K'(\ell+n) \\
&\quad + \left(K(\ell+n) - K(\ell) - \sum_{j=0}^{n-1} K'(\ell+j) \right) \\
&= K(\ell+n+1) - K(\ell) - \sum_{j=0}^{n} K'(\ell+j),
\end{aligned}$$

so $(**)$ is established.

For how many n is it true that $\xi_{\ell+n} - \eta_{k+n} < \xi_{\ell+n} - \xi_{\ell+n+1}$? By $(**)$, this condition will hold providing that

$$K(\ell+n) - K(\ell) - \sum_{j=0}^{n-1} K'(\ell+j) < K(\ell+n) - K(\ell+n+1),$$

or, equivalently,

$$\sum_{j=0}^{n-1} Q(\ell+j) < \int_{\ell}^{\ell+n+1} Q(u)\,du.$$

Since Q is decreasing, it is enough to have

$$\sum_{j=0}^{n-1} Q(\ell+j) < \sum_{j=1}^{n+1} Q(\ell+j)$$

or $Q(\ell) < Q(\ell+n) + Q(\ell+n+1)$ and so $Q(\ell) < 2Q(\ell+n+1)$ suffices. Now if $\theta_0 > 1$ and is sufficiently small, $Q(x)/Q(\theta_0 x) < 2$, provided x is large. Picking $1 < \theta < \theta_0$ and a suitably large index ℓ', we obtain the fact that if $\ell + n < \theta\ell$, $\ell > \ell'$, then we do indeed have $\xi_{\ell+n} - \eta_{k+n} < \xi_{\ell+n} - \xi_{\ell+n+1}$. To recapitulate, if an interval $[\xi_{\ell+1}, \xi_\ell)$ contains no η and ℓ is large, then the intervals $[\xi_{\ell+j+1}, \xi_{\ell+j})$ contain exactly one η for $\ell + j < \theta\ell$.

We may assume that there are infinitely many intervals $[\xi_{\ell+1}, \xi_\ell)$ not containing an η. Denote by ℓ_ν, $\nu \geq 0$, $\ell_\nu < \ell_{\nu+1}$, those indices for which $[\xi_{\ell+1}, \xi_\ell)$ has no η and $\ell_0 > \ell'$. By the results found earlier, we have $\ell_1 \geq \theta\ell_0$ and, in general, $\ell_\nu \geq \theta^\nu \ell_0$. Of course, since $\eta_0 = \xi_0$, we have $\eta_k < \xi_k$, so $\ell \geq k$. Assume first that $k \geq \ell_0$, and let $\xi_{\ell+1} \leq \eta_k < \xi_\ell$. Define μ by $\ell_{\mu+1} > \ell > \ell_\mu$. Then $\ell = k + \mu + 1 + B$ where B is the number of intervals $[\xi_{j+1}, \xi_j)$, $1 \leq j \leq \ell'$, which contain no η. Hence we have $\log\ell > \log\ell_\mu > \mu\log\theta$, and $k > \ell - (\log\theta)^{-1}\log\ell - (\ell'+1)$. By considering separately the case $\ell' \leq k < \ell_0$ for which $k = \ell - B \geq \ell - \ell'$ and the case $k \leq \ell'$, we easily obtain the first assertion of the theorem.

To check the uniformity of C, let $b < a$. $H_b(x)$ is defined by $H_b(x) = \int_x^b \frac{du}{G(u)}$, and K_b and Q_b are defined in the obvious way. For $x > 0$, $Q_b(x) = Q(x + H(b))$ and

$$\frac{Q_b(x)}{Q_b(\theta_0 x)} = \frac{Q(x+H(b))}{Q(\theta_0 x + H(b))} < \frac{Q(x+H(b))}{Q(\theta_0(x+H(b))} < 2$$

if $x + H(b) > \ell'$. Since C is determined by θ_0 and ℓ', uniformity is proved.

Finally, to check that $\xi_k \sim \eta_k$, note that K satisfies the same slowly oscillating condition as Q. If ε is small and $\theta > 1$ is suitably chosen, we get from $K(x) = \int_x^\infty Q(u)\,du$ the inequalities

$$K(x) < (1+\varepsilon)\int_x^\infty Q(\theta u)\,du = \frac{(1+\varepsilon)}{\theta}\int_{\theta x}^\infty Q(u)\,du < (1+\varepsilon)K(\theta x)$$

provided x is large. Now if $\xi_{\ell+1} \leq \eta_k < \xi_\ell$, then for large k, $\ell + 1 < k + C' \log k$, for any $C' > C$, and

$$\begin{aligned} 0 < 1 - (\eta_k/\xi_k) &\leq 1 - (\xi_{\ell+1}/\xi_k) \\ &= 1 - (K(\ell + 1)/K(k)) \\ &< 1 - (K(k + C' \log k)/K(k)). \end{aligned}$$

But $(K(\theta k)/K(k)) < \varepsilon$ if θ is suitably chosen, and the result follows.

As we mentioned, under the generality of our assumptions, we cannot expect a true second asymptotic term. Indeed, we have obtained only a nontrivial lower bound for η_k/ξ_k. Nevertheless, as crude as our method is, it seems to produce an error of the correct order of magnitude. For example, for $F(x) = \sin x$, it is known that

$$F_k(a) = (3/k)^{1/2} \left\{ 1 - \frac{3}{10} \frac{\log k}{k} + o\left(\frac{\log k}{k}\right) \right\},$$

(see [1]) if a is small. If we set $G(x) = Bx^n(1 + 0(x))$ and assume that $0 < G(y) - G(x) < y - x$ for $0 < x < y < A$, and we also assume for simplicity that $n > 2$, then a simple calculation gives $K(x) = (B(n-1)x)^{1/1-n}\{1 + 0(1/x)\}$. It follows that Q satisfies the hypothesis of Theorem 1. Hence,

$$(B(n-1)k)^{1/1-n} > \eta_k > (B(n-1)k)^{1/1-n}\{1 - D(\log k/k)\}$$

for suitable D.

Finally, for $G(x) = x^2 \exp(-1/x)$, another routine computation gives

$$\frac{1}{\log k} > \eta_k > \frac{1}{\log k} \left\{ 1 - \frac{C}{k} \right\}.$$

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