

THE DISTRIBUTION OF RADEMACHER SUMS

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ABSTRACT. We find upper and lower bounds for $\Pr(\sum \pm x_n \geq t)$, where x_1, x_2, \dots are real numbers. We express the answer in terms of the K -interpolation norm from the theory of interpolation of Banach spaces.

INTRODUCTION

Throughout this paper, we let $\varepsilon_1, \varepsilon_2, \dots$ be independent Bernoulli random variables (that is, $\Pr(\varepsilon_n = 1) = \Pr(\varepsilon_n = -1) = \frac{1}{2}$). We are going to look for upper and lower bounds for $\Pr(\sum \varepsilon_n x_n > t)$, where x_1, x_2, \dots is a sequence of real numbers such that $x = (x_n)_{n=1}^\infty \in l_2$.

Our first upper bound is well known (see, for example, Chapter II, §59 of [5]):

$$(1) \quad \Pr\left(\sum \varepsilon_n x_n > t \|x\|_2\right) \leq e^{-t^2/2}.$$

However, if $\|x\|_1 < \infty$, this cannot also provide a good lower bound, because then we have another upper bound:

$$(2) \quad \Pr\left(\sum \varepsilon_n x_n > \|x\|_1\right) = 0.$$

To look for lower bounds, we might first consider using some version of the central limit theorem. For example, using Theorem 7.1.4 of [2], it can be shown that for some constant c we have

$$\left| \Pr\left(\sum \varepsilon_n x_n > t \|x\|_2\right) - \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-s^2/2} ds \right| \leq c \left(\frac{\|x\|_3}{\|x\|_2} \right)^3.$$

Thus, for some constant c we have that if $r \leq c^{-1} (\log \|x\|_3 / \|x\|_2)^{1/2}$, then

$$\Pr\left(\sum \varepsilon_n x_n > t \|x\|_2\right) \geq c^{-1} \int_t^\infty e^{-s^2/2} ds \geq \frac{c^{-2} e^{-t^2/2}}{t}.$$

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However, we should hope for far more. From (1) and (2), we could conjecture something like

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} \inf\{\|x\|_1, t\|x\|_2\}\right) \geq c^{-1} e^{-ct^2}.$$

Actually such a conjecture is unreasonable—one should not take infimums of norms, but instead one should consider the following quantity:

$$\begin{aligned} K(x, t; l_1, l_2) &= K_{1,2}(x, t) \\ &= \inf\left\{\|x'\|_1 + t\|x''\|_2 : x', x'' \in l_2, x' + x'' = x\right\}. \end{aligned}$$

This norm is well known to the theory of interpolation of Banach spaces (see, for example [1] or [3]). For small t , this norm looks a lot like $t\|x\|_2$, but as t gets much larger, it starts to look more like $\|x\|_1$. In fact, there is a rather nice approximate formula due to T. Holmstedt (Theorem 4.1 of [3]): if we write $(x_n^*)_{n=1}^\infty$ for the sequence $(|x_n|)_{n=1}^\infty$ rearranged into decreasing order, then

$$c^{-1} K_{1,2}(x, t) \leq \sum_{n=1}^{\lfloor t^2 \rfloor} x_n^* + t \left(\sum_{n=\lfloor t^2 \rfloor + 1}^\infty (x_n^*)^2 \right)^{\frac{1}{2}} \leq K_{1,2}(x, t),$$

where c is a universal constant.

In this paper, we will prove the following result.

Theorem. *There is a constant c such that for all $x \in l_2$ and $t > 0$ we have*

$$\Pr\left(\sum \varepsilon_n x_n > K_{1,2}(x, t)\right) \leq e^{-t^2/2}$$

and

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} K_{1,2}(x, t)\right) \geq c^{-1} e^{-ct^2}.$$

An interesting example is $x = (n^{-1})_{n=1}^\infty$. Then $c^{-1} \log t \leq K_{1,2}(x, t) \leq c \log t$, and hence

$$c^{-1} \exp(-\exp(ct)) \leq \Pr\left(\sum \varepsilon_n n^{-1} > t\right) \leq c \exp(-\exp(c^{-1}t)).$$

This is quite different behavior than that which we might have expected from the central limit theorem.

We might also consider $x = (n^{-1/p})_{n=1}^\infty$, where $1 < p < 2$. This example leads us to deduce Proposition 2.1 of [7]. More involved methods allow us to rederive the results of [8] (which include the above-mentioned result from [7]). We do not go into details.

We also deduce from the following corollary.

Corollary. *There is a constant c such that for all $x \in l_2$ and $0 < t \leq \|x\|_2 / \|x\|_\infty$ we have*

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} t \|x\|_2\right) \geq c^{-1} e^{-ct^2}.$$

Proof. It is sufficient to show that there is a constant c such that if $0 < t \leq \|x\|_2 / \|x\|_\infty$, then

$$K_{1,2}(x, t) \leq t \|x\|_2 \leq cK_{1,2}(x, t).$$

The left-hand inequality follows straight away from the definition of $K_{1,2}(x, t)$. The right-hand side follows easily from Holmstedt's formula; obviously if $t < 1$, and otherwise because

$$\sum_{n=1}^{\lfloor t^2 \rfloor} x_n^* \geq \lfloor t^2 \rfloor \frac{\|x\|_2}{t} \geq \frac{t}{2} \|x\|_2. \quad \square$$

PROOF OF THEOREM

In order to prove the theorem, we will need some new norms on l_2 , and a few lemmas.

Definition. For $x \in l_2$ and $t > 0$, define the norm

$$J(x, t; l_\infty, l_2) = J_{\infty,2}(x, t) = \max \{ \|x\|_\infty, t \|x\|_2 \}.$$

Lemma 1. For $t > 0$, the spaces $(l_2, K_{1,2}, (\cdot, t))$ and $(l_2, J_{\infty,2}(\cdot, t^{-1}))$ are dual to one another, that is, for $x \in l_2$ we have

$$K_{1,2}(x, t) = \sup \left\{ \sum x_n y_n : y \in l_2, J_{\infty,2}(y, t^{-1}) \leq 1 \right\}.$$

Proof. This is elementary (see, for example Chapter 3, Exercise 1-6 of [1]). \square

Definition. For $x \in l_2$ and $t \in \mathbf{N}$, define the norm

$$\|x\|_{P(t)} = \sup \left\{ \sum_{m=1}^t \left(\sum_{n \in B_m} |x_n|^2 \right)^{1/2} \right\},$$

where the supremum is taken over all disjoint subsets, $B_1, B_2, \dots, B_t \subseteq \mathbf{N}$.

Lemma 2. If $x \in l_2$ and $t^2 \in \mathbf{N}$, then

$$\|x\|_{P(t^2)} \leq K_{1,2}(x, t) \leq \sqrt{2} \|x\|_{P(t^2)}.$$

Proof. To show the first inequality, note that we have

$$\|x\|_{P(t^2)} \leq \|x\|_1 \quad \text{and} \quad \|x\|_{P(t^2)} \leq t \|x\|_2.$$

Hence

$$\begin{aligned} K_{1,2}(x, t) &= \inf \left\{ \|x'\|_1 + t \|x''\|_2 : x' + x'' = x \right\} \\ &\geq \inf \left\{ \|x'\|_{P(t^2)} + \|x''\|_{P(t^2)} : x' + x'' = x \right\} \\ &\geq \|x\|_{P(t^2)}, \end{aligned}$$

where the last step follows by the triangle inequality.

For the second inequality, we start by using Lemma 1. For any $\delta > 0$, let $y \in l_2$ be such that

$$(1 - \delta) K_{1,2}(x, t) \leq \sum x_n y_n \quad \text{and} \quad J_{\infty,2}(y, t^{-1}) = 1.$$

Now pick numbers $n_0, n_1, n_2, \dots, n_{t^2} \in \{0, 1, 2, \dots, \infty\}$ by induction as follows: given $0 = n_0 < n_1 < \dots < n_m$, let

$$n_{m+1} = 1 + \sup \left\{ \nu : \sum_{n=n_m+1}^{\nu} |y_n|^2 \leq 1 \right\}.$$

Since $\|y\|_{\infty} \leq 1$, we have that $\sum_{n=n_m+1}^{n_{m+1}} |y_n|^2 \leq 2$. Also, as $\|y\|_2 \leq t$, it follows that $n_{t^2} = \infty$. Therefore

$$\begin{aligned} (1 - \delta) K_{1,2}(x, t) &\leq \sum x_n y_n \\ &\leq \sum_{m=1}^{t^2} \left(\sum_{n=n_{m-1}+1}^{n_m} |y_n|^2 \right)^{1/2} \left(\sum_{n=n_{m-1}+1}^{n_m} |x_n|^2 \right)^{1/2} \\ &\leq \sqrt{2} \|x\|_{p(t^2)}. \end{aligned}$$

Since this is true for all $\delta > 0$, the result follows. \square

The following lemma is due to Paley and Zygmund.

Lemma 3. *If $x \in l_2$, then given $0 < \lambda < 1$ we have*

$$\Pr \left(\sum \varepsilon_n x_n > \lambda \|x\|_2 \right) \geq \frac{1}{3} (1 - \lambda^2)^2.$$

Proof. See Chapter 3, Theorem 3 of [4]. \square

Now we proceed with the proof of the theorem. First we will show that

$$\Pr \left(\sum \varepsilon_n x_n > K_{1,2}(x, t) \right) \leq e^{-t^2/2}.$$

Given $\delta > 0$, let $x', x'' \in l_2$ be such that $x' + x'' = x$, and

$$(1 + \delta) K_{1,2}(x, t) > \|x'\|_1 + t \|x''\|_2.$$

Then

$$\begin{aligned} \Pr \left(\sum \varepsilon_n x_n > (1 + \delta) K_{1,2}(x, t) \right) &\leq \Pr \left(\sum \varepsilon_n x'_n > \|x'\|_1 \right) \\ &\quad + \Pr \left(\sum \varepsilon_n x''_n > t \|x''\|_2 \right) \\ &\leq 0 + e^{-t^2/2}, \end{aligned}$$

where the last inequality follows from equations (1) and (2) above. Letting $\delta \rightarrow 0$, the result follows.

Now we show that for some constant c we have

$$\Pr \left(\sum \varepsilon_n x_n > c^{-1} K_{1,2}(x, t) \right) \geq c^{-1} e^{-ct^2}.$$

First let us assume that $t^2 \in \mathbf{N}$. Given $\delta > 0$, let $B_1, B_2, \dots, B_{t^2} \subseteq \mathbf{N}$ be disjoint subsets such that $\bigcup_{m=1}^{t^2} B_m = \mathbf{N}$ and

$$\|x\|_{P(t^2)} \leq (1 + \delta) \sum_{m=1}^{t^2} \left(\sum_{n \in B_m} |x_n|^2 \right)^{1/2}.$$

Then

$$\begin{aligned} \Pr \left(\sum \varepsilon_n x_n > \frac{1}{2} K_{1,2}(x, t) \right) &\geq \Pr \left(\sum \varepsilon_n x_n > \frac{1}{\sqrt{2}} \|x\|_{P(t^2)} \right) \\ &\geq \Pr \left(\sum_{m=1}^{t^2} \sum_{n \in B_m} \varepsilon_n x_n \geq \frac{1}{\sqrt{2}} (1 + \delta) \sum_{m=1}^{t^2} \left(\sum_{n \in B_m} |x_n|^2 \right)^{1/2} \right) \\ &\geq \prod_{m=1}^{t^2} \Pr \left(\sum_{n \in B_m} \varepsilon_n x_n \geq \frac{1}{\sqrt{2}} (1 + \delta) \left(\sum_{n \in B_m} |x_n|^2 \right)^{1/2} \right) \\ &\geq \left(\frac{1}{3} \left(1 - \frac{1}{2} (1 + \delta)^2 \right) \right)^{t^2}, \end{aligned}$$

where the last step is from Lemma 3. If we let $\delta \rightarrow 0$, then we see that

$$\Pr \left(\sum \varepsilon_n x_n > \frac{1}{2} K_{1,2}(x, t) \right) \geq \exp \left(-(\log 12) t^2 \right).$$

This proves the result for $t^2 \in \mathbf{N}$. For $t \geq 1$, note that

$$K_{1,2}(x, t) \leq K_{1,2}(x[\lceil t \rceil]) \quad \text{and} \quad \lceil t \rceil^2 \leq 4t^2,$$

and hence the result follows (with $c = 4 \log 12$). For $t < 1$, the result may be deduced straightaway from Holmstedt's formula and Lemma 3. \square

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