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A NOTE ON WEINSTEIN'S CONJECTURE

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ABSTRACT. We prove that the contact foliation of a compact contact manifold (M, α) has at least one compact leaf in the following two cases: (i) α is a K-contact form and M is simply connected, (ii) α is C^2-close to a regular contact form. This solves the Weinstein conjecture in those particular cases.

1. THE CONJECTURE

Let S be a hypersurface in a symplectic manifold (M, Ω). There are a Riemannian metric g on M and an almost-complex structure J on M such that $\Omega(U, V) = g(U, jV)$ and $g(JU, JV) = g(U, V)$ for all vector fields U, V. The characteristic distribution X_S of S is the 1-dimensional distribution on S defined by $X_S(x) = JN_x$, where N_x is a unit outward normal vector to S at x. An integral curve of the distribution X_S is called a characteristic of S. In a famous paper [6], Rabinowitz proved that if S is a strongly star-shaped hypersurface of $\mathbf{R}^{2n}$, with its standard symplectic structure ω_0, then S has at least one closed characteristic. In [8], Weinstein conjectured that if S is simply connected and carries a contact form α such that $i*\Omega = d\alpha$ (he calls such submanifolds "hypersurfaces of contact type"), then S should have at least one closed characteristic. Here $i: S \to M$ is the inclusion map.

The conjecture has been proved by Viterbo [7] in the particular case $M = \mathbf{R}^{2n}$ with its standard symplectic form, but without the assumption that S is simply connected. Viterbo's trick is to change the problem into one of finding periodic orbits of a Hamiltonian system and use the now-familiar variational method: closed orbits correspond to critical points of an action-functional on a loop space.

Let S be a hypersurface of contact type in a symplectic manifold (M, Ω). Then $i*\Omega = d\alpha$, where $i: S \to M$ is the inclusion map. Let X_α be the Reeb vector field of α: this is the unique vector field on S such that $i(X_\alpha)\alpha = 1$ and $i(X_\alpha)d\alpha = 0$; here $i(\bullet)$ is the interior product operation. It is clear that

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the characteristic field X_S of S spans the kernel of $i * \Omega = d\alpha$. Hence, there exists a nowhere-vanishing function f on S such that $X_S(x) = f(x) \bullet X_\alpha(x)$ for all $x \in S$. Therefore, characteristics of S are just reparametrized flow lines of the Reeb field. If (S, α) is a contact manifold, the foliation F of S by integral curves of the Reeb field will be called the "contact foliation" of (S, α).

The Weinstein conjecture can be rephrased as follows: Let (S, α) be a compact, simply connected hypersurface of contact type of a symplectic manifold. Then, its contact foliation has a compact leaf.

We may forget about the embedding $S \hookrightarrow M$ and ask if the contact flow of any compact, simply connected contact manifold has a compact leaf? Posed this way, the problem becomes trivial for some types of contact structure: for instance, if α is a regular contact form.

Recall that a contact form α on S is said to be a regular contact form if each point $x \in S$ has an open neighborhood U such that the integral curves of its Reeb field X_α passing through U pass through that neighborhood only once (see [1, p. 6]). One knows that if α is a regular contact form on a compact manifold S, then there exists a smooth nowhere-vanishing function λ on S such that $X' = \lambda X_\alpha$ generates a free action of the circle S^1 on S (Boothby–Wang's theorem; see [1, p. 14]). Therefore, in that case all the leaves of the contact foliation are compact.

One would like to know what happens if one starts with a regular contact form α and adds to it a small perturbation. Since the set of contact forms is open in the set of all 1-forms, the resulting form α' is a contact form. However, α' is not necessarily a regular contact form. In the next section, we show that the contact foliation of α' has compact leaves. Finally, in §3, we show that the contact foliation F_α of a contact form α on a compact, simply connected manifold M has a compact leaf provided that there exists a Riemannian metric on M which leaves invariant the Reeb field of α. Such contact forms are called K-contact forms [1].

2. A "DEFORMATION" RESULT

Let $\pi: M \to B$ be an oriented S^1-bundle over a compact, oriented even-dimensional manifold B, and let F be an oriented 1-dimensional foliation on M with a transverse symplectic form ω, i.e., $d\omega = 0$ and $\operatorname{Ker} \omega$ generates the tangent space to the leaves of F. That is, for all $x \in M$, $\operatorname{Ker} \omega(x) = \{x \in T_xM | \omega(x)(X, \xi) = 0, \forall \xi \in T_xM\}$ is a 1-dimensional vector space isomorphic with the tangent space to the leaf of F through x.

Under the hypothesis that (i) $\omega = \pi^* \omega_0 + d\omega_1$ for some closed 2-form ω_0 on B and (ii) $\operatorname{Ker} \omega$ is C^1-close to the vertical, Ginzburg [2] has proved that the number of compact leaves of F is at least equal to the number k_π of critical points of smooth functions on M such that their critical manifolds are smooth curves whose projections are homologous to zero in B. Clearly $k_\pi \geq 2$.

Consider now a compact manifold M equipped with a regular contact form α_0, and consider a contact form α C^2-close to α_0. By Boothby–Wang's theorem [1], M is the total space of a principal S^1-bundle $\pi\colon M \to B$, where the action of S^1 on M is generated by a multiple of the Reeb field X_{α_0} of α_0. Hence the vertical direction is spanned by X_{α_0}. The contact foliation F_α of (M, α) admits $\omega = d\alpha$ as a transverse symplectic structure which satisfies hypothesis (i) in Ginzburg's theorem.

Since α and α_0 are C^2-close, $(d\alpha)^n$ and $(d\alpha_0)^n$ are C^1-close. But the Reeb field X_β of any contact form β is uniquely determined by the equation $i(X_\beta)(\beta \wedge (d\beta)^n) = (d\beta)^n$ when the dimension of the manifold is $2n+1$. Therefore if α and α_0 are C^2-close, the corresponding Reeb fields are C^1-close. Hence Ginzburg'a theorem implies the following result, a form of which was already pointed out by Ginzburg in the case where M is the unit cosphere bundle over a compact, oriented surface:

Theorem 1. *Let (M, α) be a compact manifold where the contact form α is C^2-close to some regular contact form on M. Then the contact foliation F_α of (M, α) has at least two compact leaves.*

3. K-CONTACT FOLIATIONS

A contact form α on a smooth manifold S is called a K-contact form if there exists a Riemannian metric g on S which is invariant by the Reeb field X_α of α, i.e., if $L_{X_\alpha} g = 0$, where L is the Lie derivative. The corresponding contact foliation is called a K-contact foliation.

Theorem 2. *Let (S, α) be a compact, simply connected manifold with a K-contact form α. The K-contact foliation of (S, α) has at least one compact leaf.*

Proof. Let g be a Riemannian metric on S such that $L_{X_\alpha} g = 0$, where X_α is the Reeb field of α. Monna [5] has shown that the K-contact condition is equivalent to the existence of an invariant transverse metric for the contact foliation F, i.e., that F is a Riemannian foliation with a bundle-like transverse metric in the sense of Rienhart. We refer to the excellent book of Molino [3]. For each vector field V on S, let $\overline{V}$ be the normal field to F such that $\overline{V}(x)$ is the projection on the subspace of T_xS orthogonal to $X_\alpha(x)$. Monna [5] defines a transverse metric $\overline{g}$ by the equation

$$\overline{g}(\overline{U}, \overline{V}) = g(-U + \alpha(U) \bullet X_\alpha, -V + \alpha(V) \bullet X_\alpha).$$

An easy calculation shows that indeed $\overline{g}$ is a transverse metric invariant by X_α. In fact, Monna proves that a contact foliation is transversally Riemannian with an invariant bundle-like metric if and only if the contact form is a K-contact form.

Since the kernel of $\omega = d\alpha$ is one-dimensional and spanned by X_α, the 2-form $\omega = d\alpha$ is a transverse symplectic structure for the foliation F. We are now in position to apply the geometric results of Molino on complete Riemannian foliations with transverse symplectic structures: according to Molino [4], if (S, α) is a compact, simply connected K-contact manifold and p is the dimension of the structural algebra of the complete Riemannian foliaton F (the K-contact foliation), there exists a (momentum) map $I: S \to \mathbf{R}^p$, constant on the leaves of F, such that $I(S) \subseteq \mathbf{R}^p$ is a closed convex polytope whose vertices correspond to compact leaves: in this case, these are closed curves. Since the polytope $I(S)$ necessarily has one or more vertices, the contact foliation has necessarily at least one compact leaf. □

Remarks. There are examples of K-contact forms which are not regular (see for instance [1, pp. 90–91]). On the other hand, one knows that a regular contact form is a K-contact form.

Let us finally point out that the contact form α on the 3-torus T^3 induced by the contact form $\tilde{\alpha} = \cos(2\pi x_3)\, dx_1 + \sin(2\pi x_3)\, dx_2$ on $\mathbf{R}^3$ is neither regular nor K-contact. Indeed there are closed or nonclosed leaves, according to the rationality of $\tan(2\pi x_3)$. Thus the contact foliation has compact and noncompact leaves; some of the compact leaves are not isolated. Monna pointed out to me that results of Molino prohibit this foliation from being a foliation with a bundle-like metric. Hence the contact form is not a K-contact form. Obviously it is not a regular contact form either, since the leaves are nonclosed.

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