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THE GENERAL FORM OF GREEN'S THEOREM

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ABSTRACT. Using a recently developed Perron-type integration theory, we prove a new form of Green's theorem in the plane, which holds for any rectifiable, closed, continuous curve under very general assumptions on the vector field. In particular, Cauchy's integral theorem can be deduced in its presently most general form.

1. INTRODUCTION

During the last few years, new general integration theories have been proposed by different authors in order to formulate a divergence theorem without any integrability assumptions (see [8], [10], [9], and especially [3]). Of course, one should ask how these theories lead to an improvement of Green's theorem. First results for rectifiable, simple closed, continuous curves can be found in [7] and [3].

Formulating a theorem of Green, one must certainly deal with two different kinds of assumptions, the geometrical ones and the analytical ones.

In this paper we present Green's theorem in the plane in a new form for rectifiable, closed, continuous curves and continuous vector functions, which may not fulfill a local Lipschitz condition on a large exceptional set. Thus from a geometrical point of view, our situation is as general as is possible, while our analytical assumptions concerning the given vector field are also very general. In particular, our theorem not only extends the well-known result by Shapiro [15, Theorem 5] (see also [1] and [14]), but also implies Cauchy's integral theorem in its general form including the situation of Besicovitch as stated in Saks [13].

Our result, which will be formulated in §2 and proved in §3, is based partly on a new general form of the divergence theorem for a certain Perron-type integral, taking exceptional sets into account see [3], and partly on an approximation theorem formulated by Michael [6].

In §4 we present Cauchy's integral theorem, as well as some concluding remarks related to the subject.

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We also want to mention that other authors (see [11], [12], [16], and mainly [5, Theorem 10] and [6]) proved what could be called half of Green's theorem, dealing with the occurring partial derivatives separately and not with the divergence of the given vector function. Since they have to assume the integrability of the winding number times the partial derivative, no theorem of Cauchy follows. An exception is a result by Kral and Marik [5, Theorem 9], where no integrability assumptions are needed. This is obtained by replacing the double integral by an iterated integral, but again, Cauchy's theorem cannot be deduced.

2. Green's theorem

We begin by introducing our assumptions, as well as some notations. After this we will recall some properties of the *-integral, which will be used in the proof of our theorem. Concerning *-integration and the divergence theorem, the reader is referred to a paper by Jurkat [3], where this new integration theory is developed.

This section closes with the presentation of our form of Green's theorem.

Assumptions and notations.

(i) Let γ be a rectifiable, closed, continuous curve in $\mathbf{R}^2$. We denote by Γ the image of γ in $\mathbf{R}^2$ and by $w_\gamma(z)$ the winding number of γ with respect to $z \in \mathbf{R}^2 - \Gamma$. For $z \in \Gamma$, we set $w_\gamma(z) = 0$, so that $G = \{z \in \mathbf{R}^2 \colon w_\gamma(z) \neq 0\}$ is an open set in $\mathbf{R}^2$. Its closure is denoted by $\overline{G}$. Furthermore we set

$$[\pm N, w] = \begin{cases} N, & \text{if } w \geq N \\ w, & \text{if } |w| \leq N \\ -N, & \text{if } w \leq -N \end{cases}$$

for $N \in \mathbf{N}$, $w \in \mathbf{R}$, and

$$G_p = \{z \in \mathbf{R}^2 \colon w_\gamma(z) = p\}, \; p \in \mathbf{Z},$$

$$G_N^* = \{z \in \mathbf{R}^2 \colon w_\gamma(z) \geq N\}, \quad G_{-N}^* = \{z \in \mathbf{R}^2 \colon w_\gamma(z) \leq -N\}.$$

(ii) Let $\mathbf{v} = (f, g)$ be a continuous function: $G \cup \Gamma \to \mathbf{R}^2$. We denote by L the set of points $z_0 \in G$, where

$$\limsup_{z \to z_0} \frac{\|\mathbf{v}(z) - \mathbf{v}(z_0)\|}{\|z - z_0\|} < \infty.$$

For those points of G, where $\mathbf{v}$ is totally differentiable we set $\operatorname{div} \mathbf{v} = \partial f/\partial x + \partial g/\partial y$ $(z = (x, y))$. At all other points of $G \cup \Gamma$ we set $\operatorname{div} \mathbf{v} = 0$, so that $\operatorname{div} \mathbf{v}$ is a uniquely defined function: $G \cup \Gamma \to \mathbf{R}$.

(iii) We assume that $G - L$ is a countable union of sets of finite one-dimensional outer Hausdorff measure, in short, a σ_1-finite set.

Some facts about *-integration. By $\mathscr{A}$ we denote the class of all compact sets $A \subseteq \mathbf{R}^n$ $(n \in \mathbf{N})$ whose boundary ∂A is of finite $(n-1)$-dimensional outer Hausdorff measure $\mathscr{H}$.

Extension of the Lebesgue integral. Let $f\colon A \to \mathbf{R}$ be a function on $A \in \mathscr{A}$. If a finite Lebesgue integral ${}^{L}\!\int_A f$ exists, then f is *-integrable on A and

$$\overset{*}{\int_A} f = \overset{L}{\int_A} f.$$

Additivity property of the **-integral.* Let A, B, $C \in \mathscr{A}$, $C = A \cup B$ with disjoint interiors $\overset{\circ}{A}$ and $\overset{\circ}{B}$ (nonoverlapping union). If a function $f\colon C \to \mathbf{R}$ is *-integrable on A and B, then f is *-integrable on C and

$$\overset{*}{\int_C} f = \overset{*}{\int_A} f + \overset{*}{\int_B} f.$$

The Divergence Theorem. Let $\mathbf{v}\colon A \to \mathbf{R}^n$ be a bounded vector function on $A \in \mathscr{A}$, satisfying the following two conditions:

(1) $\mathbf{v}$ is continuous on A except for a set of $\mathscr{H}$-measure zero.
(2) $A - L$ is a σ_{n-1}-finite set.

Then $\operatorname{div}\mathbf{v}$ is *-integrable on A and

$$\int_{\partial A} \mathbf{v}\cdot\mathbf{n}_A \, d\mathscr{H} = \overset{*}{\int_A} \operatorname{div}\mathbf{v}.$$

Here $\mathbf{n}_A$ denotes the exterior normal as defined in [2].

Formulation of our theorem.

Theorem. *Under the assumptions* (i), (ii) *and* (iii) *we have*

$$\int_\gamma (f\,dy - g\,dx) = \lim_{N\to\infty} \overset{*}{\int_{\overline{G}}} [\pm N, w_\gamma] \operatorname{div}\mathbf{v}, \tag{1}$$

where all occurring integrals and the limit exist.

3. Proof of Green's theorem

In the first part of the proof, we follow Michael [6] in treating the left-hand side of (1).

Observe, that G is bounded, and its boundary is contained in Γ, which has finite one-dimensional Hausdorff measure. Similar statements are true for G_p $(p \neq 0)$ and $G^*_{\pm N}$.

We extend f, g to all of $\mathbf{R}^2$ as continuous functions with bounded support, so that they become uniformly continuous, and introduce the modulus of continuity

$$h(\delta) = \sup_{\|z_2 - z_1\| \le \delta} \|\mathbf{v}(z_2) - \mathbf{v}(z_1)\|, \qquad \delta > 0.$$

For $k \in \mathbf{N}$ we consider the net S_k consisting of all squares $[\mu/2^k, (\mu+1)/2^k] \times [\nu/2^k, (\nu+1)/2^k]$ with μ, $\nu \in \mathbf{Z}$ and mesh-size $\eta_k = \sqrt{2}/2^k$.

Then we define the sequence k_n $(n \in \mathbf{N})$ inductively by the condition that k_n be the first $k \in \mathbf{N}$ with $h(\eta_k) < \frac{1}{n^3}$ so that $k_n \nearrow$ and $n^2 h(\eta_{k_n}) \to 0$.

Given $N \in \mathbf{N}$, we consider all squares K_i from S_{k_N} with $K_i \subseteq G$ and form the corresponding cycle

$$\gamma_N = \sum_i [\pm N, w_\gamma(K_i)] \partial^+ K_i, \tag{2}$$

where $\partial^+ K_i$ denotes the cycle formed by the boundary of K_i with natural parametrization and positive orientation.

From Michael's proof [6, p. 4], we see that

$$\int_\gamma (f\,dy - g\,dx) = \lim_{N\to\infty} \int_{\gamma_N} (f\,dy - g\,dx). \tag{3}$$

This follows from the assumptions (i) and (ii), without (iii).

In the second part of the proof we use the divergence theorem as stated before and apply it to the squares K_i occurring in (2):

$$\begin{gathered} \int_{\partial^+ K_i} (f\,dy - g\,dx) = {}^*\!\!\int_{K_i} \operatorname{div} \mathbf{v}, \\ [\pm N, w_\gamma(K_i)] \int_{\partial^+ K_i} (f\,dy - g\,dx) = {}^*\!\!\int_{K_i} [\pm N, w_\gamma] \operatorname{div} \mathbf{v}, \end{gathered} \tag{4}$$

since $w_\gamma(z)$ is constant on K_i. By summing we obtain

$$\int_{\gamma_N} (f\,dy - g\,dx) = {}^*\!\!\int_{K(N)} [\pm N, w_\gamma] \operatorname{div} \mathbf{v}, \quad \text{where } K(N) = \cup K_i. \tag{5}$$

Next we consider all squares $\widetilde{K}_i$ from S_{k_N} which intersect Γ and observe that $\overline{G} \subseteq K(N) \cup \widetilde{K}(N)$ with $\widetilde{K}(N) = \cup \widetilde{K}_i$. Since $\overline{G}$ is the nonoverlapping union of all $\overline{G}_p$ $(0 < |p| < N)$ together with $\overline{G}^*_N$ and $\overline{G}^*_{-N}$, it follows by additivity that the integrals

$$ {}^*\!\!\int_{\overline{G}\cap\widetilde{K}_i} [\pm N, w_\gamma] \operatorname{div} \mathbf{v} = {}^*\!\!\int_{\overline{G}^*_N \cap \widetilde{K}_i} + {}^*\!\!\int_{\overline{G}^*_{-N} \cap \widetilde{K}_i} + \sum_{0<|p|<N} {}^*\!\!\int_{\overline{G}_p \cap \overline{K}_i} \tag{6}$$

exist, since $[\pm N, w_\gamma]$ is a.e. constant in the integrals on the right, so that these integrals exist by the divergence theorem.

Adding up the integrals on the left side of (6) we infer that

$$ {}^*\!\!\int_{\overline{G}} [\pm N, w_\gamma] \operatorname{div} \mathbf{v} = {}^*\!\!\int_{K(N)} + {}^*\!\!\int_{\overline{G}\cap\widetilde{K}(N)},$$

where all integrals exist.

It remains to prove that

$$ {}^*\!\!\int_{\overline{G}\cap\widetilde{K}(N)} [\pm N, w_\gamma] \operatorname{div} \mathbf{v} \to 0, \qquad (N \to \infty). \tag{7}$$

In the last part of the proof, we rewrite the integrals occurring on the right-hand side of (6) by using the divergence theorem as follows:

$$ {}^*\!\!\int_{\overline{G}_p\cap\widetilde{K}_i} [\pm N, w_\gamma] \operatorname{div} \mathbf{v} = p \int_{\partial(\overline{G}_p\cap\widetilde{K}_i)} \mathbf{v}\cdot\mathbf{n}_{\overline{G}_p\cap\widetilde{K}_i}\, d\mathscr{H}.$$

By subtracting from $\mathbf{v}$ one of its values on $\widetilde{K}_i$, we obtain for the modulus of that integral the estimate

$$\leq Nh(\eta_{k_N})\mathscr{H}[\partial(\overline{G}_p \cap \widetilde{K}_i)].$$

Since $\partial(\overline{G}_p \cap \widetilde{K}_i) \subseteq (\partial\widetilde{K}_i) \cup (\mathring{\widetilde{K}}_i \cap \partial\overline{G}_p)$ and the number of all $\widetilde{K}_i$ is at most $\leq c(\gamma)/\eta_{k_N}$ (with some positive constant $c(\gamma)$ depending only on γ, as is well known), we get by adding the estimates above

$$\left|{}^*\!\!\int_{\overline{G}\cap\widetilde{K}(N)} [\pm N, w_\gamma] \operatorname{div} \mathbf{v}\right|$$
$$\leq Nh(\eta_{k_N}) \sum_i \left(2N\mathscr{H}(\partial\widetilde{K}_i) + \mathscr{H}(\mathring{\widetilde{K}}_i \cap \partial\overline{G}^*_N) + \mathscr{H}(\mathring{\widetilde{K}}_i \cap \partial\overline{G}^*_{-N}) \right.$$
$$\left. + \sum_{0<|p|<N} \mathscr{H}(\mathring{\widetilde{K}}_i \cap \partial\overline{G}_p) \right).$$

Because $\mathscr{H}(\partial\widetilde{K}_i) \leq c_1\eta_{k_N}$ (with an absolute constant c_1) and $\partial\overline{G}_p \subseteq \Gamma$, $\partial\overline{G}^*_N \subseteq \Gamma$, $\partial\overline{G}^*_{-N} \subseteq \Gamma$, it follows with another absolute constant c_2

$$\left|{}^*\!\!\int_{\overline{G}\cap\widetilde{K}(N)} [\pm N, w_\gamma] \operatorname{div} \mathbf{v}\right| \leq c_2 c(\gamma) N^2 h(\eta_{k_N}) + c_2\mathscr{H}(\Gamma) N^2 h(\eta_{k_N}).$$

Here the right-hand side tends to zero by construction, which completes the proof.

4. Concluding remarks

(i) The divergence theorem was only used in parts two and three of the proof. Clearly any improvement of the divergence theorem will lead to an improvement of Green's theorem.

(ii) In our proof we use essentially the continuity of the given vector field. The question in how far this requirement may be relaxed in the general geometrical setting will be investigated in a subsequent paper.

Nevertheless, we want to point out that, in case of a rectifiable, simple closed, continuous curve γ, we can allow an exceptional set of $\mathscr{H}$-measure zero for the continuity, provided that $\mathbf{v}$ is bounded (see [3]).

(iii) While other authors introduced exceptional sets with respect to the differentiability of the vector function, on our exceptional set a local Lipschitz condition may fail. Obviously our situation is much more general and seems to have been overlooked by many authors.

(iv) Trivially, if a finite Lebesgue integral ${}^L\!\!\int_G w_\gamma \operatorname{div} \mathbf{v}$ exists, we may replace the right-hand side of (1) by this expression. Since ${}^L\!\!\int_G |w_\gamma| < \infty$, this is particularly true for a bounded divergence on $\overline{G}$.

(v) Let us rewrite our theorem as follows.

Under the assumptions (i), (ii), and (iii), we have

$$(1)' \qquad \int_\gamma (f\,dy - g\,dx) = \lim_{N\to\infty} \overset{*}{\int}_{K(N)} [\pm N, w_\gamma] \operatorname{div} \mathbf{v},$$

where all occurring integrals and the limit exist.

This observation is important, since in order to prove $(1)'$ the divergence theorem is only needed for squares. Thus any improvement of the divergence theorem for rectangles leads to an improvement of $(1)'$. One should compare this remark with results like [4], where the general theorem is based on its validity for rectangles.

(vi) Finally we deduce from our theorem Cauchy's integral theorem in its general form:

Let γ, Γ, G be as in §2 and $f(z)$ be a continuous complex function on $G \cup \Gamma$, which satisfies the condition

$$\limsup_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} < \infty$$

at each point z_0 of G except at most those of a σ_1-finite set. Then

$$\int_\gamma f(z)\,dz = 0,$$

provided that the Cauchy-Riemann equations hold almost everywhere on G.

Surprisingly, no such theorem has been formulated before.

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