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UNIVERSAL MAPS AND SURJECTIVE CHARACTERIZATIONS OF COMPLETELY METRIZABLE LC^n-SPACES

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ABSTRACT. We construct an n-dimensional completely metrizable $AE(n)$-space $P(n, \tau)$ of weight $\tau \geq \omega$ with the following property: for any at most n-dimensional completely metrizable space Y of weight $\leq \tau$ the set of closed embeddings $Y \to P(n, \tau)$ is dense in the space $C(Y, P(n, \tau))$. It is also shown that completely metrizable LC^n-spaces of weight $\tau \geq \omega$ are precisely the n-invertible images of the Hilbert space $\ell_2(\tau)$.

INTRODUCTION

Let $\mathscr{Y}$ be a class of completely metrizable spaces. A space $X \in \mathscr{Y}$ is said to be *strongly $\mathscr{Y}$-universal* if for any space $Y \in \mathscr{Y}$ the set of closed embeddings $Y \to X$ is dense in the space $C(Y, X)$ of all continuous maps from Y to X endowed with the limitation topology (a stronger version of this notion under the same name was introduced in [BM]). This property is very important in the theory of manifolds modelled on certain model spaces. Let us recall the corresponding results:

(i) If $\mathscr{R}$ is the class of all metrizable compacta, then X is homeomorphic to the Hilbert cube Q iff X is a strongly $\mathscr{R}$-universal AE-compactum $[T_1]$;

(ii) if $\mathscr{R}_n$ is the class of all at most n-dimensional metrizable compacta, then X is homeomorphic to Menger's universal n-dimensional compactum M_n^{2n+1} [E] iff X is a strongly $\mathscr{R}_n$-universal $AE(n)$-compactum (for $n = 0$, [Br]; for $n \geq 1$, [B]);

(iii) if $\mathscr{M}_\tau$ is the class of all completely metrizable spaces of weight $\leq \tau$, $\tau \geq \omega$, then X is homeomorphic to the Hilbert space $\ell_2(\tau)$ iff X is a strongly $\mathscr{M}_\tau$-universal AE-space [T_2].

(iv) if $\mathscr{M}_{0,\tau}$ is the class of all zero-dimensional completely metrizable spaces of weight $\leq \tau$, $\tau \geq \omega$, then X is homeomorphic to the Baire space $B(\tau)$ iff X is strongly $\mathscr{M}_{0,\tau}$-universal (for $\tau = \omega$ [AU]; for $\tau > \omega$, [St]).

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The main purpose of this paper is to show the existence of a strongly $\mathscr{M}_{n,\tau}$-universal $AE(n)$-space, where $\mathscr{M}_{n,\tau}$ is the class of all at most n-dimensional completely metrizable spaces of weight $\leq \tau$, $\tau \geq \omega$. Let us note that strongly $\mathscr{M}_{n,\omega}$-universal $AE(n)$-spaces were constructed by the first author in [C_3]. There are many reasons to hope that the following problems have affirmative solutions:

Problem. Are any two strongly $\mathscr{M}_{n,\tau}$-universal $AE(n)$-spaces homeomorphic? In particular, is any strongly $\mathscr{M}_{n,\omega}$-universal $AE(n)$-space homeomorphic to Nobeling's universal n-dimensional space N_n^{2n+1} ?

The second part of this paper is devoted to surjective characterizations of completely metrizable LC^n-spaces. Similar characterizations in the class of metrizable compacta were earlier obtained by Hoffman [H_2] and Dranishnikov [D] (see also [C_4], where the class of Polish spaces is considered).

1. Preliminaries

All spaces considered are metrizable and maps continuous. By dimension dim we mean covering dimension. A metrizable space X is an *absolute* (*neighborhood*) *extensor* in dimension n (briefly, $X \in A(N)E(n)$) if for any at most n-dimensional metrizable space Y and any closed subspace A of it each map $f: A \to X$ can be extended to the whole of Y (respectively, to a neighborhood of A in Y). It is well known that for $n > 0$, $X \in A(N)E(n)$ iff $X \in LC^{n-1} \cap C^{n-1}$ (respectively, $X \in LC^{n-1}$). Note also that any metrizable space, and hence any completely metrizable space, is an $AE(0)$. (The argument can be made as follows: Let $\dim Y = 0$ and $A \subset Y$ be closed. Let X be metrizable and $f: A \to X$ be a map. It is well known that there is a retraction $r: Y \to A$. Then $f \circ r$ is the desired extension of f.) The notion of *n-soft map* between compacta was introduced by Schepin [S]. Later Chigogidze [C_1] extended it to the class of all Tychonov spaces. Below we use the following definition of this notion: a map $f: X \to Y$ between metrizable spaces is called *n-soft* if for any at most n-dimensional paracompact space Z, any closed subspace A of it and any two maps $g: Z \to Y$, $h: A \to X$ with $f \circ h = g|A$, there exists a map $k: Z \to X$ such that $f \circ k = g$ and $k|A = h$.

2. Universal maps

Lemma 2.1. *Let $f: X \to Y$ be an n-soft map between metrizable spaces. Suppose $\dim X \leq n$ and Y is a complete absolute extensor for the class of all metrizable spaces. Then X is complete.*

Proof. Consider the Stone–Čech compactification βX of X. Denote by Z the space obtained from βX by means of making the points of $\beta X - X$ isolated. This space is shown to be paracompact in the proof of [P, Lemma 2].

Claim. $\dim Z \leq n$. Since Z is normal it suffices to extend to Z an arbitrary map $g: F \to S^n$ from a closed subset F of Z into the n-dimensional sphere.

The case $F \subseteq Z - X$ is trivial. Suppose now that $F \cap X \neq \emptyset$. Since $\dim X \leq n$, there exists an extension $g_1 : F \cup X \to S^n$ of g. Observe that $F \cup X$ is closed in Z. Hence we can extend g_1 to a map g_2 from Z into the $(n+1)$-dimensional disk B^{n+1}. Put $H = g_2^{-1}(B^{n+1} - \{b\})$, where $b \in B^{n+1} - S^n$. Fix a retraction $r: (B^{n+1} - \{b\}) \to S^n$. Clearly, H is clopen in Z; so there exists a map $g_3: Z \to S^n$ extending the composition $r \circ g_2: H \to S^n$. Obviously, $g_3|F = g$. The claim is proved.

Since Y is metrizable and is an AE for metrizable spaces, then Y is an absolute retract for metrizable spaces. Further, complete metrizable implies Čech complete. Being metrizable, Y is a paracompact p-space, so using [P, Fact 6 and Corollary 1(b)], we conclude that Y is an absolute extensor for the class of collectionwise normal spaces. Take a map $h: Z \to Y$ such that $h|X = f$. It follows by the n-softness of f that there exists a retraction from Z onto X. Now, by arguments of Przymusinski ([P, the proof of Lemma 2]), X is complete.

Lemma 2.2 (for $\tau = \omega$, [C_2]). *Let $0 \leq n < \omega \leq \tau$. Then there exist an n-dimensional completely metrizable space X of weight τ and an n-soft map $f: X \to \ell_2(\tau)$.*

Proof. By [C_2, Theorem 5], we can suppose that $\tau > \omega$. Let d_1 be any metric on $\ell_2(\tau)$. Fix a completely zero-dimensional (with respect to d_1) surjection $g: \ell_2(\tau) \to Y$, where Y is a separable metrizable space [AP]. By an application of [C_2, Theorem 5.1] (see also the remarks following it) there exist an at most n-dimensional separable space Z and an n-soft map $h: Z \to Y$. Let X be a fibered product (pullback) of the spaces $\ell_2(\tau)$ and Z with respect to the maps g and h. Denote by $f: X \to \ell_2(\tau)$ and $p: X \to Z$ the corresponding canonical projections. Let d_2 be any metric on Z. Clearly, the map p is completely zero-dimensional with respect to the metric $d = (d_1^2 + d_2^2)^{1/2}$ [AP, Chapter 6, §3, Lemma 4]. Hence $\dim X \leq \dim Z \leq n$. It is easy to see that $\omega(X) = \tau$. Observe also that the n-softness of h implies the n-softness of f. Since $\ell_2(\tau)$ contains a copy of the n-dimensional cube I^n and f is an n-soft map, the space X contains a copy of I^n too. Thus $\dim X = n$. By Lemma 2.1, X is completely metrizable.

Corollary 2.3 (for $\tau = \omega$, [C_3]). *Let $0 \leq n < \omega \leq \tau$. Then for every (completely) metrizable space Y of weight τ there exist an at most n-dimensional (completely) metrizable space Z of weight τ and an n-soft map $g: Z \to Y$.*

Proof. Embed Y into $\ell_2(\tau)$ as a (closed) subspace and consider the map $f: X \to \ell_2(\tau)$ from Lemma 2.2. Put $Z = f^{-1}(Y)$ and $g = f|Z$.

Definition 2.4 (for $\tau = \omega$, [C_3]). Let $0 \leq n < \omega \leq \tau$. A map $f: X \to Y$ is said to be (n, τ)-full if for any map $g: Z \to Y$ from any at most n-dimensional completely metrizable space Z of weight $\leq \tau$ there exists a closed embedding $h: Z \to X$ such that $f \circ h = g$.

Definition 2.5 (for $\tau = \omega$, [C$_3$]). Let $0 \le n < \omega \le \tau$. A map $f\colon X \to Y$ is called strongly (n, τ)-universal if for any open cover $\mathscr{U}$ of X, any at most n-dimensional completely metrizable space Z of weight $\le \tau$ and any map $g\colon Z \to X$ there exists a closed embedding $h\colon Z \to X$ $\mathscr{U}$-close to g with $f \circ h = f \circ g$. We shall also say that a space X is strongly (n, τ)-universal if the constant map $X \to *$ is strongly (n, τ)-universal in the above sense.

Lemma 2.6 (for $\tau = \omega$, [C$_3$]). *Let $0 \le n < \omega \le \tau$ and $S = \{X_k, p_k^{k+1}, \omega\}$ be an inverse sequence consisting of completely metrizable spaces X_k of weight $\le \tau$ and n-soft, (n, τ)-full projections p_k^{k+1}. Then the limit projection $p_0\colon X \to X_0$, where $X = \lim S$, is strongly (n, τ)-universal.*

Proof. Equip X with the metric $d(\{x_k\}, \{y_k\}) = \max_k d_k(x_k, y_k)$, where d_k is a metric for X_k with $d_k \le 2^{-k}$, $k \in \omega$. If suffices to show that, given a completely metrizable space Y with $\dim Y \le n$ and $\omega(Y) \le \tau$ and maps $f\colon Y \to X$, $\alpha\colon X \to (0, 1)$, there is a closed embedding $g\colon Y \to X$ with $p_0 \circ g = p_0 \circ f$ and $d(f(y), g(y)) \le \alpha(f(y))$ for each $y \in Y$.

For each $k \in \omega$ fix a closed embedding $i_{k+1}\colon X_{k+1} \to \ell_2(\tau)$. Since $\ell_2(\tau) \in AE$ there is a map $h\colon (\ell_2(\tau))^2 \times [0, \infty) \to \ell_2(\tau)$ such that $h(a, b, t) = a$ for $t \le 1$ and $h(a, b, t) = b$ for $t \ge 2$. By Corollary 2.3, for each $k \in \omega$ there is an n-soft map $q_{k+1}\colon Z_{k+1} \to X_k \times \ell_2(\tau)$, where Z_{k+1} is a completely metrizable space with $\dim Z_{k+1} \le n$ and $\omega(Z_{k+1}) = \tau$. The n-softness of the projection p_k^{k+1} implies the existence of a $r_{k+1}\colon Z_{k+1} \to X_{k+1}$ with $p_k^{k+1} \circ r_{k+1} = \pi_k \circ q_{k+1}$ and $r_{k+1}|A_{k+1} = (p_k^{k+1} \Delta i_{k+1})^{-1} \circ q_{k+1}|A_{k+1}$, where

$$A_{k+1} = q_{k+1}^{-1}((p_k^{k+1} \Delta i_{k+1})(X_{k+1})) \quad \text{and} \quad \pi_k\colon X_k \times \ell_2(\tau) \to X_k$$

denotes the natural projection. Put $g_0 = p_0 \circ f$. By our assumption, the projection p_0^1 of the spectrum S is (n, τ)-full. Hence there exists a closed embedding $j_1\colon Y \to X_1$ such that $p_0^1 \circ j_1 = g_0$. Consider now the map

$$g_0 \Delta h(i_1 \circ p_1 \circ f \Delta i_1 \circ j_1 \Delta 2\alpha \circ f)\colon Y \to X_0 \times \ell_2(\tau).$$

Since $\dim Y \le n$ and the map q_1 is n-soft there is a map $s_1\colon Y \to Z_1$ such that $g_0 \Delta h(i_1 \circ p_1 \circ f \Delta i_1 \circ j_1 \Delta 2\alpha \circ f) = q_1 \circ s_1$. We define a map $g_1\colon Y \to X_1$ by the formula $g_1 = r_1 \circ s_1$. Note that $p_0^1 \circ g_1 = g_0$. If $y \in Y$ and $\alpha(f(y)) \le 2^{-1}$, then $g_1(y) = p_1(f(y))$. (Use the fact that $g_1(y) = r_1(s_1(y))$, show that $s_1(y) \in A_1$, and then use the above formula for $q_1 \circ s_1$.)

Let us construct a map $g_2\colon Y \to X_2$. By our assumption, the projection p_1^2 is (n, τ)-full. Hence there is a closed embedding $j_2\colon Y \to X_2$ such that $p_1^2 \circ j_2 = g_1$. Since $\dim Y \le n$ and the map q_2 is n-soft there is a map $s_2\colon Y \to Z_2$ such that $q_2 \circ s_2 = g_1 \Delta h(i_2 \circ p_2 \circ f \Delta i_2 \circ j_2 \Delta 2^2 \alpha \circ f)$. Put $g_2 = r_2 \circ s_2$. As above, $p_1^2 \circ g_2 = g_1$. Note that if $y \in Y$ and $\alpha(f(y)) \le 2^{-2}$, then $g_2(y) = p_2(f(y))$; if $\alpha(f(y)) \ge 2^{-1}$, then $g_2(y) = j_2(y)$.

Let us suppose that for each i, $2 \leq i \leq k$, we have already constructed maps $g_i \colon Y \to X_i$ and closed embeddings $j_i \colon Y \to X_i$ satisfying the following conditions:

$(1)_i$ $p^i_{i-1} \circ g_i = g_{i-1}$;

$(2)_i$ $p^i_{i-1} \circ j_i = g_{i-1}$;

$(3)_i$ if $y \in Y$ and $\alpha(f(y)) \leq 2^{-i}$, then $g_i(y) = p_i(f(y))$;

$(4)_i$ if $y \in Y$ and $\alpha(f(y)) \leq 2^{-i+1}$, then $g_i(y) = j_i(y)$.

Let us construct a map $g_{k+1} \colon Y \to X_{k+1}$ and a closed embedding $j_{k+1} \colon Y \to X_{k+1}$ with the desired properties. Choose an arbitrary closed embedding $j_{k+1} \colon Y \to X_{k+1}$ with $p^{k+1}_k \circ j_{k+1} = g_k$ (the existence of j_{k+1} follows from the (n, τ)-fullness of p^{k+1}_k). As above there is a map $s_{k+1} \colon Y \to Z_{k+1}$ such that $q_{k+1} \circ s_{k+1} = g_k \Delta h(i_{k+1} \circ p_{k+1} \circ f \Delta i_{k+1} \circ j_{k+1} \Delta 2^{k+1} \alpha \circ f)$. We put $g_{k+1} = r_{k+1} \circ s_{k+1}$. The verification of the conditions $(1)_{k+1}$—$(4)_{k+1}$ is left to the reader.

It follows from the conditions $(1)_k$, $k \in \omega$, that the diagonal product $g = \Delta\{g_k \colon k \in \omega\}$ maps Y into X and satisfies the following equalities: $p_k \circ g = g_k$, $k \in \omega$. In particular, we have $p_0 \circ g = g_0 = p_0 \circ f$.

Let $y \in Y$. If $\alpha(f(y)) \in [2^{-k-1}, 2^{-k}]$, then $2^i \alpha(f(y)) \leq 1$ for $i \leq k$; hence

$$p_i(f(y)) = g_i(y) \quad \text{for } i \leq k$$

and

$$\begin{aligned} d(f(y)g(y)) &= \max\{d_i(p_i(f(y)), g_i(y)) \colon i = k+1, k+2, \dots\} \\ &\leq 2^{-k-1} \leq \alpha(f(y)). \end{aligned}$$

In order to show that g is a closed embedding it suffices to use the corresponding arguments from the proof of [C_2, Lemma 7.11].

Theorem 2.7 (for $\tau = \omega$, [C_3]). *Let $0 \leq n < \omega \leq \tau$. Then there exist an n-dimensional completely metrizable space $P(n, \tau)$ of weight τ and a strongly (n, τ)-universal n-soft map $f(n, \tau) \colon P(n, \tau) \to \ell_2(\tau)$.*

Proof. Put $X_0 = l_2(\tau)$. By Corollary 2.3, there exists at most an n-dimensional completely metrizable space X_{k+1} of weight τ and an n-soft map $h_{k+1} \colon X_{k+1} \to X_k \times \ell_z(\tau)$. Put $p^{k+1}_k = \pi_k \circ h_{k+1}$, where $\pi_k \colon X_k \times \ell_2(\tau) \to X_k$ is the natural projection, $k \in \omega$. So we get an inverse sequence $S = \{X_k, p^{k+1}_k\}$ consisting of at most n-dimensional completely metrizable spaces of weight τ and (n, τ)-full n-soft projections. Put $P(n, \tau) = \lim S$ and $f(n, \tau) = p_0$. By Lemma 2.6, the map $f(n, \tau)$ is strongly (n, τ)-universal. Obviously, this map is n-soft. Since $\dim X_k \leq n$ for each $k \geq 1$, we have $\dim P(n, \tau) \leq n$ [N]. The inverse inequality $\dim P(n, \tau) \geq n$ follows from the strong (n, τ)-universality of $f(n, \tau)$. Finally observe that $P(n, \tau)$ is a complete metrizable space of weight τ. This completes the proof.

Corollary 2.8 (for $\tau = \omega$, [C_3]). *Let $0 \leq n < \omega \leq \tau$. The space $P(n, \tau)$ is an n-dimensional strongly (n, τ)-universal completely metrizable $AE(n)$-space of weight τ.*

Proof. $P(n, \tau)$ is an $AE(n)$-space as an n-soft preimage of $\ell_2(\tau)$.

Remark 2.9. The space $P(n, \tau)$ has the following property: for every open subspace U of $P(n, \tau)$ and any at most n-dimensional completely metrizable space X of weight $\leq \tau$ there exists an embedding $h: X \to U$ such that $h(X)$ is closed in $P(n, \tau)$. Indeed, consider a constant map $g: X \to \{p\}$, where p is an arbitrary point of U, and the open cover $\mathscr{U} = \{U, P(n, \tau) - \{p\}\}$ of $P(n, \tau)$. By Definition 2.5, there is a closed embedding $h: X \to P(n, \tau)$ $\mathscr{U}$-close to g. Clearly, $h(X) \subset U$. In particular, we have $\omega(U) = \tau$ for every open subset U of $P(n, \tau)$. Hence, $P(0, \tau)$ is homeomorphic to the Baire space $B(\tau)$ (see [St]).

Remark 2.10. A. Wasko proved, in [W], that for every $n \geq 0$ and every $\tau \geq \omega$ there exists an n-dimensional completely metrizable space $X_{n,\tau}$ of weight τ such that every at most n-dimensional completely metrizable space of weight $\leq \tau$ is embedded in $X_{n,\tau}$ as a closed subset (for $\tau = \omega$ this was proved earlier by the first author, see [C_1, Corollary 3]). Recently E. Pol, [Po], strengthened this result of A. Wasko by proving that for every n-dimensional completely metrizable space X of weight $\leq \tau$ the set of all embeddings of X onto a closed subset of $S(\tau)^\omega$ contained in $K_n(\tau)$ is residual in the space $C(X, S(\tau)^\omega)$. Here $S(\tau)$ is the τ-star-space and $K_n(\tau)$ denotes Nagata's universal n-dimensional space.

3. Surjective characterizations of LC^n-spaces

Definition 3.1 [H_2]. A space X is said to be in the class $AE(n, m)$, where $0 \leq n \leq m \leq \infty$, if for any metrizable space Z with $\dim Z \leq m$, any closed subspace A of it with $\dim A \leq n$, any map $f: A \to X$ can be extended to the whole of Z. If f can be extended only to a neighborhood of A in Z, we get a definition of the class $ANE(n, m)$.

Lemma 3.2. *For every $n \in \omega$ the following equalities are true in the class of all metrizable spaces: $A(N)E(n+1) = A(N)E(n, n+1)$.*

In the class of all metrizable compacta this lemma was proved by Dranishnikov (see [D, Lemma 3.1]). The same proof remains valid in the general case.

Lemma 3.3. *If $0 \leq n < m \leq \infty$, then the following equalities are true in the class of all metrizable spaces: $A(N)E(n+1) = A(N)E(n, m)$.*

Proof. Since $n + 1 \leq m$, then $A(N)E(n, m) \subset A(N)E(n, n+1)$; thus, $A(N)E(n, m) \subset A(N)E(n+1)$ follows from Lemma 3.2. Suppose X is a metrizable $AE(n+1)$-space. Let Z be a metrizable space with $\dim Z \leq m$, A a closed subspace of Z with $\dim A \leq n$ and f a map from A to X. Take a metrizable AE-space, say Y, of dimension $\leq n+1$ containing A as a closed subspace ([K], see also [C_4] for Polish spaces and [Ts] for completely metrizable spaces). Now choose a map $k: Y \to X$ such that $k|A = f$. This is possible because, by Lemma 3.2, $X \in AE(n, n+1)$. Since $Y \in AE$, there exists a map $g: Z \to Y$ such that $g|A = id$. Then $k \circ g$ is an extension of f. The inclusion $ANE(n+1) \subset ANE(n, m)$ follows from the same arguments.

Definition 3.4 [H_1]. A map $f: X \to Y$ is called n-invertible if for any at most n-dimensional metrizable space Z and any map $g: Z \to Y$ there exists a map $h: Z \to X$ such that $g = f \circ h$.

Obviously, every n-invertible map is a surjection. For a metrizable compactum X it is known [D], [H_2] that X is an $AE(n+1)$ iff X is an n-invertible image of Q. It is also true [D] that the class of all metrizable $AE(n)$-compacta coincides with the class of n-invertible images of the n-dimensional universal Menger compactum m_n^{2n+1}. The first author [C_4] gave similar characterizations of $AE(n+1)$ and $AE(n)$ in the class of Polish spaces. In the case of metrizable spaces of uncountable weight we know only the following facts: (i) the classical characterization of completely metrizable spaces of weight τ as open images of the Baire space $B(\tau)$; (ii) a metrizable space X of weight τ is Čech-complete iff X is a 0-invertible image of $B(\tau)$ [V].

Below we give similar characterizations of metrizable $AE(n)$-spaces of arbitrary weight.

Definition 3.5 [C_4]. A map $f: X \to Y$ is said to be inductively n-soft if there exists a closed subspace Z of X such that the restriction $f|Z: Z \to Y$ is n-soft.

Theorem 3.6. *Let X be a metrizable space of weight $\tau \geq \omega$. Then for every $n \in \omega$ the following conditions are equivalent:*

(i) $X \in AE(n+1)$ (*respectively,* $X \in ANE(n+1)$);
(ii) *X is an inductively n-soft image of an AE* (*respectively, of an ANE*);
(iii) *X is an n-invertible image of an AE* (*respectively, of an ANE*);

Proof. We shall prove only the global variant. The local one follows from the same arguments.

(i) $\to$ (ii). By Corollary 2.3, there exist an at most n-dimensional metrizable space Y of weight τ and an n-soft map $g: Y \to X$. Embed Y into a metrizable AE-space Z as a closed subspace. By Lemma 3.3, there exists an extension $h: Z \to X$ of g. Clearly, h is inductively n-soft.

(ii) $\to$ (iii). This implication is trivial, because any n-soft map is n-invertible.

(iii) $\to$ (i). Let Z be an AE-space and $f: Z \to X$ be an n-invertible map. In view of Lemma 3.2 it suffices to show that $X \in AE(n, n+1)$. Let B be any at most $(n+1)$-dimensional metrizable space, A a closed subspace of B with $\dim A \leq n$ and g a map from A to X. Since f is n-invertible, there exists a map $h: A \to Z$ such that $f \circ h = g$. Take any extension $k: B \to Z$ of h. Then the map $f \circ k$ is an extension of g.

Let us consider the proof of Theorem 3.6. If X is a complete metrizable space of weight τ, then the space Y is also complete, so we can suppose that Z is the space $\ell_2(\tau)$. Thus, the following theorem is true.

Theorem 3.7. *Let X be a completely metrizable space of weight $\tau \geq \omega$. Then for every $n \in \omega$ the following conditions are equivalent:*

(i) $X \in AE(n+1)$ *(respectively, $X \in ANE(n+1)$);*
(ii) *X is an inductively n-soft image of $\ell_2(\tau)$ (respectively, of an open subspace of $\ell_2(\tau)$);*
(iii) *X is an n-invertible image of $\ell_2(\tau)$ (respectively, of an open subset of $\ell_2(\tau)$).*

The proof of the following result is analogous to the proof of Theorem 3.6.

Theorem 3.8. *Let X be a completely metrizable space of weight $\tau \geq \omega$. Then for every $n \in \omega$ the following conditions are equivalent:*

(i) $X \in AE(n)$ *(respectively, $X \in ANE(n)$);*
(ii) *X is an inductively n-soft image of $P(n, \tau)$ (respectively, of an open subset of $P(n, \tau)$);*
(iii) *X is an n-invertible image of $P(n, \tau)$ (respectively, of an open subset of $P(n, \tau)$).*

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