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UNIQUE CONTINUATION THEOREMS FOR SOME PARABOLIC OPERATORS

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ABSTRACT. We prove a unique continuation theorem for a class of differential operators containing the heat operator.

Suppose $j_1, \ldots, j_n$ and j are positive integers and let E be the differential operator on $\mathbf{R}^n$ whose symbol is

$$\sigma(\xi) = -\left(\sum_{l=1}^{n} \xi_l^{2j_l}\right)^j .$$

The purpose of this paper is to prove a unique continuation theorem for the operator $L = \partial/\partial t - E$ on $\mathbf{R}^{n+1}$.

Theorem. *Let* $r = \sum_{l=1}^n (2jj_l)^{-1} + 1$. *Suppose* $u \in C_0^\infty(\mathbf{R}^{n+1})$ *satisfies the differential inequality*

$$|Lu(x, t)| \le |V(x, t)u(x, t)| \tag{1}$$

for some $V \in L^r(\mathbf{R}^{n+1})$. *Then* $u \equiv 0$.

If L is the heat operator $\partial/\partial t - \Delta$, then $r = n/2+1$. In this case our theorem yields a conclusion analogous to [3, Corollary 1], although that result for the Schrödinger operator $i\partial/\partial t + \Delta$ requires only that u vanish in a half-space. Our method of proof is also similar to that of [3] (and [2]) in that we deduce our unique continuation theorem from a Carleman estimate which is a consequence of a uniform Sobolev inequality. But while the Sobolev inequalities of [2] and [3] are based on restriction theorems for the Fourier transform, ours depends only on Young's inequality and an elementary estimate. This may reflect the fact that our operators L are "less singular" than those treated in [2] and [3]. Our main task, then, will be to establish the following result.

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Lemma 1. *With L and r as above, fix p and q with $1 < p < q < \infty$ and $1/p - 1/q = 1/r$. There is a constant $C = C(L, p)$ such that for any $\lambda < 0$ and any $u \in C_0^\infty(\mathbf{R}^{n+1})$ we have*

$$\|u\|_q \leq C\|(L-\lambda)u\|_p. \tag{2}$$

Combining the observation

$$e^{\lambda t} Lu = (L-\lambda)v \qquad \text{if } v = e^{\lambda t}u \tag{3}$$

with (2) yields a Carleman estimate

$$\|e^{\lambda t}u\|_q \leq C(p)\|e^{\lambda t}Lu\|_p, \qquad u \in C_0^\infty(\mathbf{R}^{n+1}). \tag{4}$$

Now let W^p be the Sobolev space on $\mathbf{R}^{n+1}$ defined by the norm

$$\left\| \left[\left(\sqrt{1+\tau^2} - \sigma(\xi) \right) \hat{u}(\xi, \tau) \right]^\vee \right\|_p.$$

A short argument similar to the proof of [2, Corollary 3.1] shows that (4) implies the following: if $1 < p < r$ and $u \in W^p$ satisfies (1) for some $V \in L^r(\mathbf{R}^{n+1})$, then $u \equiv 0$ if u vanishes on a half-space $\{(x, t): t \geq t_0\}$. This clearly yields our theorem.

The proof of Lemma 1 depends on an elementary result given below as Lemma 2. To state it we require some notation. If $\beta = (\beta_1, \dots, \beta_n)$ is an n-tuple of positive numbers, define

$$|\beta| = \sum_{l=1}^n \beta_l$$

and, for $t > 0$ and $x \in \mathbf{R}^n$,

$$t^\beta x = (x_1 t^{\beta_1}, \dots, x_n t^{\beta_n}), \qquad \frac{x}{t^\beta} = \left(\frac{1}{t}\right)^\beta x.$$

Lemma 2. *Suppose β is as above and $f \in L^{1+1/|\beta|}(\mathbf{R}^n)$. Define*

$$K(x, t) = \begin{cases} t^{-|\beta|} f(\frac{x}{t^\beta}) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then $K \in L^{1+1/|\beta|, \infty}(\mathbf{R}^{n+1})$.

Proof. The measure in $\mathbf{R}^n$ of the set

$$\left\{ x: t^{-|\beta|} \left| f\left(\frac{x}{t^\beta}\right) \right| > s \right\}$$

is equal to

$$|\{t^\beta y: |f(y)| > st^{|\beta|}\}| = t^{|\beta|}|\{y: |f(y)| > st^{|\beta|}\}| \doteq t^{|\beta|} D(st^{|\beta|}).$$

Thus the measure in $\mathbf{R}^{n+1}$ of the set $\{(x, t): |K(x, t)| > s\}$ is

$$\int_0^\infty t^{|\beta|} D(st^{|\beta|})\, dt = |\beta|^{-1} s^{-1-1/|\beta|} \int_0^\infty u^{1/|\beta|} D(u)\, du.$$

This last integral is finite if (and only if) $f \in L^{1+1/|\beta|}(\mathbf{R}^n)$.

Proof of Lemma 1. Let r' be the conjugate index of r. We will construct fundamental solutions K_λ for the operators $L - \lambda$ on $\mathbf{R}^{n+1}$ such that

$$\|K_\lambda\|_{r',\infty} \le C$$

for some $C = C(L)$ and all $\lambda < 0$. Since $1/p + 1/r' = 1/q + 1$, it will follow from Young's convolution inequality for weak L^p (see the comment on [4, p. 121]) that

$$\|K_\lambda * w\|_q \le C(L, p)\|w\|_p$$

for $w \in C_0^\infty(\mathbf{R}^{n+1})$. Taking $w = (L-\lambda)u$ then yields (2). Now if $K(x, t)$ is a fundamental solution for L, then (3) shows that $K_\lambda(x, t) = e^{\lambda t}K(x, t)$ is a fundamental solution for $L - \lambda$. If K vanishes for $t < 0$ and if $\lambda < 0$, it follows that

$$\|K_\lambda\|_{r',\infty} \le \|K\|_{r',\infty}.$$

Thus it is sufficient to find a fundamental solution K for L which is in $L^{r',\infty}(\mathbf{R}^{n+1})$ and vanishes for $t < 0$. Let $\beta = ((2jj_1)^{-1}, \ldots, (2jj_n)^{-1})$ and let f be the rapidly decreasing function on $\mathbf{R}^n$ satisfying $\hat{f}(\xi) = e^{\sigma(\xi)}$. Define K as in Lemma 2. Since $1 + 1/|\beta| = r'$, it follows from Lemma 2 that $K \in L^{r',\infty}(\mathbf{R}^{n+1})$. The equation $\sigma(t^\beta\xi) = t\sigma(\xi)$ and a change of variables show that, for $t > 0$,

$$\widehat{K}(\cdot, t)(\xi) = e^{t\sigma(\xi)}, \qquad \xi \in \mathbf{R}^n.$$

It follows from this and an argument analogous to [1, Proof of Theorem 4.6, p. 195] that K is a fundamental solution for L on $\mathbf{R}^{n+1}$.

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