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TEICHMÜLLER-TYPE ANNIHILATORS OF L^1-ANALYTIC FUNCTIONS

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ABSTRACT. Characterizations are obtained for some classes of Teichmüller-type L^∞ functions that are orthogonal to all polynomials in the unit disk.

1. INTRODUCTION

For $U = \{z : |z| < 1\}$ we let

$$B(U) = \left\{ g : g(z) \text{ is holomorphic in } U, \|g\| = \iint_U |g(z)|\, dx\, dy < \infty \right\}.$$

A function $\nu(z)$ of class $L^\infty(U)$ belongs to the *annihilating class* $N(U)$ if [1]

$$\iint_U \nu(z) g(z)\, dx\, dy = 0 \quad \text{for all } g \in B(U). \tag{1.1}$$

We shall be concerned with functions $\nu(z)$ of *Teichmüller* type; that is, $\nu(z)$ is of the form

$$\nu(z) = \frac{\overline{\varphi(z)}}{|\varphi(z)|}, \qquad z \in U, \tag{1.2}$$

where $\varphi(z)$ is single-valued analytic in U.

The background of the problems that we deal with is the following question. Let $Q(z)$ be a given quasiconformal self map of U. One asks whether there exists a Teichmüller mapping $T(z)$, that is, a quasiconformal mapping of U with complex dilatation of the form

$$\frac{T_{\bar z}}{T_z} = k \frac{\overline{\varphi(z)}}{|\varphi(z)|},$$

where k is a constant, such that $T|_{\partial u} = Q|_{\partial U}$ and whether a mapping T of this form is unique. In the simplest special case, $Q(z) = I(z) = z$, we have the trivial solution, $k = 0$, φ arbitrary. We are forced to disallow the trivial solution if T is subject to side conditions that dictate that T differs from I,

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e.g., as in the condition for Teichmüller's Verschiebungssatz, where $T(0)$ is specified distinct from 0. If k is small, the condition that $T|_{\partial U} = I|_{\partial U}$ is, up to terms of order $o(k)$, known [1] to be equivalent to (1.1). Thus the problem of finding those φ for which a function ν of Teichmüller type belongs to $N(U)$ provides a first-order model for studying the uniqueness question when the boundary restriction is the identity and $k \neq 0$. Typically, a side condition on T involving a specification of T or a derivative of T at a point of U, leads, if k is minimized, to a pole of φ at the point. We shall therefore allow φ to have poles in U. Our principal result (Theorem 3.1) is stated under the assumption that $\varphi(z)$ is meromorphic in $\overline{U}$. It will be clear from the method, however, that some weakening of this assumption would have been possible.

Let Ω be a bounded region, $\nu \in L^\infty(\Omega)$. The integral

$$u(z) = -\frac{1}{\pi}\iint_\Omega \nu(\zeta)\frac{d\xi\, d\eta}{\zeta - z}, \qquad \zeta = \xi + i\eta, \tag{1.3}$$

is well defined in the complex plane $\mathbf{C}$ and holomorphic in $\mathbf{C}\backslash\overline{\Omega}$. Moreover, as is well known [4], $u(z)$ is uniformly continuous in $\mathbf{C}$, with a modulus of continuity that is $O(\delta\log(1/\delta))$. Also, $\partial u/\partial \bar{z} = \nu$ in the sense of distributions in Ω. Now if Ω contains an open disk V in which we have $\nu(z) = \overline{\varphi(z)}/|\varphi(z)|$, where $\varphi(z)$ is holomorphic and nonvanishing in V, then $\nu \in C^\infty(V)$ and so, by "elliptic regularity theory," $u \in C^\infty(V)$. In our case this is also an immediate consequence of the usual Pompeiu representation of $\overline{\Phi(z)}/\Phi'(z)$, where $\Phi(z)$ is a single-valued branch of $\int\sqrt{\varphi(z)}\,dz$ in V. For later purposes we shall also require the fact that $\partial u/\partial z$ is locally L^2 in Ω, an immediate consequence of (1.3) and $\nu \in L^2(\Omega)$.

Note that, since the polynomials are dense in $B(U)$ in the L^1 norm, it suffices that (1.1) hold for $g(z) = z^n$, $n = 0, 1, 2, \ldots$, to imply that $\nu \in N(U)$. Alternatively [1], it is necessary and sufficient that $u(z)$, as defined by (1.3) with $\Omega = U$, vanishes when $|z| = 1$.

2. The functions $\Lambda(z)$, $\lambda(z)$

Let $\varphi(z)$ be holomorphic, but not identically vanishing in a region Ω. We define $\Lambda(z)$, $\lambda(z)$ as follows. Let S_φ be the set of *critical points* (zeroes or poles) of φ in Ω. Starting with a branch of $\Phi(z) = \int\sqrt{\varphi(z)}\,dz$ in any simply connected subregion of $\Omega\backslash S_\varphi$, $\Phi(z)$ can be continued along any path in $\Omega\backslash S_\varphi$. Accordingly, we determine

$$\Lambda(z) = \overline{\Phi(z)} - \Phi'(z)u(z) \tag{2.1}$$

along any such path, where $u(z)$ is defined by (1.3).

Since $\Lambda_{\bar{z}} = \overline{\Phi'(z)} - \Phi'(z)(\overline{\varphi(z)}/|\varphi(z)|) = 0$ in $\Omega\backslash S_\varphi$ and Λ is C^∞, it follows that $\Lambda(z)$ is analytic (although not necessarily single-valued) in $\Omega\backslash S_\varphi$. We now define

$$\lambda(z) = \Lambda'(z)^2, \qquad z \in \Omega\backslash S_\varphi.$$

By the *order* of a critical point we understand an integer m, such that $|m|$ equals the order of the zero or pole depending on whether m is respectively positive or negative. If $m = 0$, the point is a regular point.

Theorem 2.1. *If $\varphi(z)$ is meromorphic in Ω and has critical set S_φ, then $\lambda(z)$ is also meromorphic in Ω. All poles of $\lambda(z)$ lie in S_φ. If $\varphi(z)$ has a critical point of order m at z_0 then $\lambda(z)$ has a critical (or regular) point of order $\geq m-2$ at z_0.*

Proof. We note that, locally,

$$-\Lambda'(z) = \frac{\partial}{\partial z}(\Phi' u) = \Phi' u_z + \Phi'' u, \qquad z \in \Omega\backslash S_\varphi. \tag{2.2}$$

Continuation of $\Phi'(z)$ around a closed curve in $\Omega\backslash S_\varphi$ results in multiplication of the function element by ± 1, and the same factor acts on $\Phi''(z)$. Thus $\lambda(z)$ is single-valued in $\Omega\backslash S_\varphi$. In terms of $\varphi(z)$,

$$\lambda(z) = \varphi(z)u_z^2 + \varphi'(z)u(z)u_z + \frac{\varphi'(z)^2}{4\varphi(z)}u(z)^2, \qquad z \in \Omega\backslash S_\varphi.$$

Suppose $\varphi(z)$ has a critical point, say at $z = 0$. Then $\lambda(z)$ has an isolated singularity at 0. To investigate the nature of the singularity, suppose first that φ has a zero of order $m \geq 1$ at 0. Then, on some neighborhood W of 0, we have $|\Phi'(z)| \leq c_1|z|^{m/2}$ and $|\Phi''(z)| \leq c_2|z|^{(m/2)-1}$ (where c_j here and in the following denote positive constants), and so, from (2.2) in view of the boundedness of u,

$$|\lambda(z)| \leq c_3(|z|^m|u_z|^2 + |z|^{m-2}), \qquad z \in W. \tag{2.3}$$

It follows that $\lambda \in L^1(W)$, and therefore that λ is either regular or has at worst a simple pole at $z = 0$. Now, the assertion to be proved is that $\lambda(z) = O(|z|^{m-2})$ near 0. If this is false then, since λ is meromorphic at 0, we must have $|\lambda(z)| \geq c_4|z|^{m-3}$ and hence, from (2.3), $|u_z|^2 \geq c_5|z|^{-3}$ for some $c_5 > 0$ contradicting that $u_z \in L^2(W)$. The proof in the case φ has a pole at 0 is nearly identical: if φ has a pole of order $|m|$ at 0 then (2.3) still holds, with $m < 0$. This readily implies that $|\lambda(z)|^\varepsilon$ is integrable on W for some $\varepsilon > 0$. From the subharmonicity of $|\lambda|^\varepsilon$ on the disk $\{z: |z - z_0| < |z_0|\}$ with small $|z_0|$ we then get $|\lambda(z_0)| \leq C|z_0|^{-2/\varepsilon}$, so again λ has at worst a pole at z_0. The remainder of the proof is nearly identical with that for the case $m > 0$ and we omit the details. □

Note that Theorem 2.1 allows for the occurrence of a simple pole of $\lambda(z)$ at a simple zero of $\varphi(z)$. As shown in §5 this possibility may actually take place.

3. Annihilation when φ is meromorphic in $\overline{U}$

If $\Lambda(z)$ has the form (2.1) in a half-neighborhood ω of a boundary point of U and if $\Phi(z)$ is sufficiently nice on a boundary arc and $u(z)$ vanishes there, then $\Phi(z)$ can be continued across the arc with the help of (2.1) to $\omega^* = \{1/\bar{z}: z \in \omega\}$. In particular,

Lemma 3.1. *Let Γ denote an open arc on ∂U all points of which are accessible boundary points of a region ω, $\omega \subset U$. Suppose $\nu(z) = \overline{\Phi'(z)}/\Phi'(z)$, $z \in \omega$, where $\Phi(z)$ is holomorphic in ω and continuous in $\omega \cup \Gamma$. If $\nu \in N(U)$, then Φ has a holomorphic extension to $\omega \cup \Gamma \cup \omega^*$, given by $\Phi(z) = \overline{\Lambda(1/\bar{z})}$, $z \in \omega^*$.*

We can apply this to obtain

Theorem 3.1. *Suppose $\varphi(z)$ is meromorphic in the closed unit disk $\overline{U}$. A necessary condition that $\overline{\varphi}/|\varphi|$ be in $N(U)$ is that $\varphi(z)$ be a rational function satisfying the relation*

$$z^4 \varphi(z) = \overline{\lambda(1/\bar{z})}, \qquad |z| > 1. \tag{3.1}$$

Proof. Let S_φ denote the set of critical points of φ in $\overline{U}$. Let Γ be an open arc on ∂U containing no points of S_φ, and let ω be a simply connected subregion of U containing no points of S_φ such that all points of Γ are accessible boundary points of ω. In ω there exists a single-valued branch of $\Phi(z)$. Let

$$u(z) = -\frac{1}{\pi} \iint_U \frac{\overline{\varphi(\zeta)}}{|\varphi(\zeta)|} \frac{d\xi\, d\eta}{\zeta - z}, \qquad z \in U,$$

and define $\Lambda(z)$ in $\overline{U} \backslash S_\varphi$ by (2.1). $\Lambda(z)$ is also holomorphic in ω, and, by Lemma 3.1, Φ and Λ can be extended to satisfy

$$\Phi(z) = \overline{\Lambda(1/\bar{z})}, \qquad z \in \omega \cup \Gamma \cup \omega^*.$$

Therefore, $\Phi'(z)$ and $\Lambda'(z)$ can be extended to satisfy

$$\Phi'(z) = -z^{-2}\overline{(\Lambda'(1/\bar{z})}, \qquad z \in \omega \cup \Gamma \cup \omega^*.$$

Squaring, we obtain (3.1) and conclude that φ is rational. This completes the proof. □

4. $\sqrt{\varphi(z)}$ HOLOMORPHIC

In the special case when $\varphi(z)$ has a single-value square root $f(z)$ which is holomorphic in U and meromorphic in $\overline{U}$, Theorem 3.1 can be replaced by a more explicit condition, as follows.

Theorem 4.1. *Suppose $f(z)$ is a function, not identically 0, holomorphic in U and meromorphic in $\overline{U}$. A necessary and sufficient condition that $\bar{f}/f \in N(U)$ is that $f(z) = F'(z)$, $z \in U$, where $F(z)$ is a nonconstant rational function with the following properties:*

(a) *All poles of $F(z)$ are on ∂U; that is, F is holomorphic for $|z| < 1$ and for $|z| > 1$, including at ∞.*

(b) *If $f(z_0) = 0$, $f'(z_0) = 0, \ldots, f^{(s)}(z_0) = 0$, for some $z_0 \in U$ and $s \geq 0$, then $F(z) - F(z_0)$ has a zero of multiplicity at least $s+1$ at $z = 1/\bar{z}_0$.*

Proof. In terms of the notation of the previous sections, we have $\overline{\varphi}/|\varphi| = \overline{\Phi}'/\Phi'$; let $f(z) = \Phi'(z)$, $F(z) = \Phi(z)$, where $F(z)$ is holomorphic in U. We apply Lemma 3.1 with $\omega = U$, where Γ is any open arc on ∂U containing no poles of f. By Lemma 3.1, $F(z)$ extends as a rational function to $\widehat{\mathbf{C}}$, satisfying

$$F(z) = \overline{\Lambda(1/\bar{z})}, \qquad 1 < |z| \leq \infty.$$

The remainder of the assertion follows from Theorem 5 of [3]. □

An implicit consequence of Theorem 4.1 is that f cannot have any first-order poles on ∂U if $\bar{f}/f$ belongs to $N(U)$. Following similar results of Ortel [2], this fact can be formulated quantitatively as follows:

Theorem 4.2. *Suppose $f(z)$ is measurable in U and, for some $z_0 \in \partial U$ and $c \neq 0$,*

$$\lim_{z \to z_0} (z - z_0) f(z) = c.$$

If $\|\nu\|_\infty \leq 1$, and if $\nu(z) = \overline{f(z)}/f(z)$ in a neighborhood of z_0 in U, then

$$\sup\left\{\left|\iint_U \nu(z) g(z)\, dx\, dy\right| \Big/ \|g\| : g \in B(U)\right\} = 1.$$

Proof. It suffices to show that if $\nu(z) \in L^\infty(U)$, and if, for some $z_0 \in \partial U$,

$$\lim_{z \to z_0} (\overline{z - z_0}/z - z_0)\nu(z) = 1, \tag{4.1}$$

then

$$\sup\left\{\left|\iint_U \nu(z) g(z)\, dx\, dy\right| \Big/ \|g\| : g \in B(U)\right\} \geq 1. \tag{4.2}$$

If we map U conformally onto the right-half plane H, transforming ν and g so that $\nu(z)\overline{dz}/dz$ and $g(z)\,dz^2$ remain, respectively, invariant, we can replace U by H and z_0 by 0 in (4.1) and (4.2). In H, consider

$$g_n(z) = \frac{z^{1/n}}{\pi n (z+1) z^2}, \qquad n = 2, 3, \ldots,$$

where $z^{1/n}$ denotes the principal branch. Since $\max(1, |z|) \leq |z+1| \leq \rho + 1$ when $z \in H$, $|z| \leq \rho$, we have

$$\frac{\rho^{1/n}}{1+\rho} \leq \iint_{\{|z|<\rho\} \cap H} |g_n(z)|\, dx\, dy \leq \rho^{1/n},$$

$$\iint_{\{|z|>\rho\} \cap H} |g_n(z)|\, dx\, dy \leq \frac{1}{(n-1)\rho^{1-1/n}}, \qquad n = 2, 3, \ldots.$$

It follows that

$$\lim_{n \to \infty} \iint_H |g_n(z)|\, dx\, dy = 1.$$

Near $z = 0$, when n is large, $n\pi g_n(z)$ behaves approximately like z^{-2} in H, while, by (4.1), given $\varepsilon > 0$,

$$|\nu(z) - (z/\bar{z})| < \varepsilon \quad \text{for } |z| < \delta,\ z \in H,$$

if $\delta > 0$ is sufficiently small. As a consequence,

$$\lim_{n\to\infty} \left| \iint_H \nu(z) g_n(z)\, dx\, dy \right| = 1,$$

completing the proof. □

Remark. In view of the Hahn–Banach Theorem, we have an equivalent dual form of Theorem 4.2; namely,

The distance in $L^\infty(U)$ from ν to $N(U)$ is not less than 1.

5. Examples

Example 5.1. $\varphi(z) = z(1-az)^2$, $z \in U$, where $|a| < 1$. According to Theorem 2.1, we can expect $\lambda(z)$ to have at worst a pole of order one at $z = 0$. As the result of a computation we obtain

$$u(z) = \frac{2}{1-az}\left[\frac{|z|^3}{3z^2} - \frac{a^2}{3} - \frac{\bar{a}|z|^5}{5z^3} + \frac{a^2|a|^2}{5}\right], \qquad |z| < 1,$$

and, by (2.1),

$$\lambda(z) = \left(\frac{1}{3} - \frac{|a|^2}{5}\right)^2 \frac{a^4}{z}, \qquad |z| < 1.$$

Thus, $\lambda(z)$ has a simple pole when $a \neq 0$, but when $a = 0$, $\lambda(z)$ is identically zero.

Example 5.2. $\lambda(z) = \frac{1}{z}$, $|z| < 1$. We have

$$\frac{\overline{\varphi(z)}}{|\varphi(z)|} = \frac{z}{|z|} = e^{i\theta}, \qquad z = re^{i\theta},\ 0 \leq r < 1.$$

Therefore

$$\frac{\overline{\varphi(z)}}{|\varphi(z)|} \in N(U).$$

So, we can use this to illustrate Theorem 3.1. A computation gives

$$u(z) = \begin{cases} -2(1-|z|), & |z| < 1, \\ 0, & |z| > 1, \end{cases}$$

and by (2.1) we obtain $\lambda(z) = z^{-3}$, $z \in U$. Condition (3.1) is evidently satisfied.

Example 5.3 [3]. $F(z)$ schlicht in U, $F(z)$ rational, $F(U)$ = unbounded region. Concrete examples are obtained when $F(U)$ is a halfplane, or a plane slit along a radial line, or the "exterior" of a parabolic region.

In all cases the hypotheses of Theorem 4.1 are satisfied, with condition (b) satisfied vacuously, since $F'(z)$ does not vanish in U.

Example 5.4. The purpose of this example is to show that hypothesis (b) of Theorem 4.1 may hold nonvacuously. To construct an example we replace $U = \{|z| < 1\}$ by $H_1 = \{\operatorname{Im} w > 0\}$. If $w(z)$ is a conformal map of U onto H_1 and if we define $\widetilde{F}(w) = F(z)$, then $\overline{\widetilde{F}'(w)}/\widetilde{F}'(w)$ will belong to $N(H_1)$ if and only if $\overline{f(z)}/f(z)$ belongs to $N(U)$. Conditions (a) and (b) become

($\tilde{\text{a}}$) All poles of $\widetilde{F}(w)$ are on $\mathbf{R}$.

($\tilde{\text{b}}$) If $\widetilde{F}'(w_0) = \widetilde{F}''(w_0) = \cdots = \widetilde{F}^{(s+1)}(w_0) = 0$ for some $w_0 \in H_1$, $s \geq 0$, then $\widetilde{F}(w) - \widetilde{F}(w_0)$ has a zero of multiplicity at least $s+1$ at $w = \overline{w_0}$.

We define

$$\widetilde{F}(w) = \frac{1}{w} - \frac{1+i}{w-1} - \frac{1-i}{w+1},$$

obtaining

$$\begin{aligned} w^2(w-1)^2(w+1)^2\widetilde{F}'(w) &= w^4 + 4iw^3 + 4w^2 - 1 \\ &= (\alpha - 1)(\alpha^3 + 5\alpha^2 + \alpha + 1), \end{aligned}$$

where $w = i\alpha$. Since $\alpha^3 + 5\alpha^2 + \alpha + 1$ has no roots with $\operatorname{Re}\alpha > 0$, $\widetilde{F}'(w_0) = 0$ in H_1 if and only if $w_0 = i$; moreover, $\widetilde{F}''(i) \neq 0$. Since $\widetilde{F}(-i) = \widetilde{F}(i)$, both ($\tilde{\text{a}}$) and ($\tilde{\text{b}}$) are satisfied.

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