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THE LIE ALGEBRA ASSOCIATED TO THE LOWER CENTRAL SERIES OF A LINK GROUP AND MURASUGI'S CONJECTURE

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ABSTRACT. The Chen-Milnor presentation can be used to determine the Lie Algebra associated to the lower central series of the fundamental group of a link in the 3-sphere S^3 in many interesting cases. We use this fact to obtain new and simpler proofs of unpublished results of Maeda on a conjecture of Murasugi in the sharpened form of Massey and Traldi.

STATEMENT OF RESULTS

Let L be a link in S^3 with m components and let $G = \pi_1(S^3 - L)$ be the group of the link. Let $F = F(m)$ be the free group on $x_1, \dots, x_m$ and let F_n denote the nth term of the lower central series of F. By a result of Milnor [9, Theorem 4] there are, for each integer $q \geq 1$, elements $w_1^{(q)}, \dots, w_m^{(q)} \in F$ such that $w_i^{(q+1)} \equiv w_i^{(q)}$ modulo F_{q+1} and such that G/G_{q+1} has the presentation

$$\langle x_1, \dots, x_m : [x_1, w_1^{(q)}] = \cdots [x_m, w_m^{(q)}] = 1,\ F_{q+1} = 1\rangle$$

where $[x, y] = x^{-1}y^{-1}xy$. Moreover, for any $q \geq 1$, any one of the relators $r_i^{(q)} = [x_i, w_i^{(q)}]$ is redundant. Let $R^{(q)}$ be the normal subgroup of F generated by $r_1^{(q)}, \dots, r_m^{(q)}$.

Let $\mathfrak{L} = \mathfrak{L}(F) = \bigoplus_{q\geq 1} \mathfrak{L}_q(F)$ be the Lie algebra associated to the lower central series of F. If ξ_i is the image of x_i in $\mathfrak{L}_1$ then $\mathfrak{L}$ is the free Lie algebra over $\mathbf{Z}$ on $\xi_1, \dots, \xi_m$. Let $\mathfrak{g} = \mathfrak{L}(G)$ be the Lie algebra associated to the lower central series of G. If $\mathfrak{g}^{(q)} = \mathfrak{L}(F/R^{(q)}F_{q+1})$ then $\mathfrak{g}^{(q)} = \bigoplus_{1\leq i\leq q} \mathfrak{g}_i = \mathfrak{L}(G/G_{q+1})$. Since $\mathfrak{g}^{(q+1)} = \mathfrak{g}^{(q)} \oplus \mathfrak{L}_{q+1}(F/R^{(q+1)}F_{q+2})$, we see that $\mathfrak{g} = \bigoplus_{q\geq 1} \mathfrak{L}_q(F/R^{(q)}F_{q+1})$. Hence $\mathfrak{g} = \mathfrak{L}/\mathfrak{R}$ where $\mathfrak{R} = \bigoplus_{q\geq 1} \mathfrak{R}_q$ with $\mathfrak{R}_q = \operatorname{Ker}(\mathfrak{L}_q(F) \to \mathfrak{L}_q(F/R^{(q)}F_{q+1}))$. The ideal $\mathfrak{R}$ is in general very difficult to determine. However, generators for it can be given if the relators $r_i^{(q)}$ satisfy a certain *independence condition*, which we now describe.

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Without loss of generality we may assume that $r_i^{(q)} \neq 1$ for $1 \leq i \leq s$ and q sufficiently large while $r_i^{(q)} = 1$ for $i > s$, $q \geq 1$. Then, for $1 \leq i \leq s$ there are integers d_i, q_i such that for $q \geq q_i$ we have $r_i^{(q)} \in F_{d_i}$ but $\notin F_{d_i+1}$. For $q \geq q_i$ the image of $r_i^{(q)}$ in $\mathfrak{L}_{d_i}$ is independent of q and is denoted by ρ_i. Also one can assume that ρ_s is a linear combination of $\rho_1, \dots, \rho_{s-1}$, cf. [10, p. 295]. Let $\mathfrak{r}$ be the ideal of $\mathfrak{L}$ generated by $\rho_1, \dots, \rho_{s-1}$. We have $\mathfrak{r} \subseteq \mathfrak{R}$ but this inclusion is in general proper. If $U = U(\mathfrak{L}/\mathfrak{r})$ is the enveloping algebra of $\mathfrak{L}/\mathfrak{r}$ then $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a U-module via the adjoint representation of $\mathfrak{L}/\mathfrak{r}$. In [7] we show that $\mathfrak{R} = \mathfrak{r}$ if the following condition holds:

(I) $\mathfrak{L}/\mathfrak{r}$ *is a free* $\mathbf{Z}$*-module and* $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ *is a free* U*-module on the images of* $\rho_1, \dots, \rho_{s-1}$.

Following [2] and [5], we call a sequence $\rho_1, \dots, \rho_{s-1}$ of elements of $\mathfrak{L}$ *inert* if condition (I) holds. For examples where this condition applies cf. [2], [4]. Anick [2] has shown that (I) holds if and only if, for $n \geq 1$, the nth homogeneous component of $\mathfrak{L}/\mathfrak{r}$ is a free $\mathbf{Z}$-module of rank a_n, where a_n is given by (A):

$$\text{(A)} \qquad \prod_{n \geq 1} (1 - t^n)^{a_n} = 1 - mt + \sum_{1 \leq j < s} t^{d_j}.$$

If (I) holds then the inverse of the above power series is the Poincaré series of the graded algebra $gr(\mathbf{Z}[G]) = \bigoplus_{n \geq 0} I^n/I^{n+1}$, where I is the augmentation ideal of the group ring $\mathbf{Z}[G]$. This follows from the fact that, in this case, $gr(\mathbf{Z}[G])$ is the enveloping algebra of $\mathfrak{L}(G)$, cf. [7].

Theorem 1. *The following are equivalent*:
(1) *Property* (I);
(2) *For* $n \geq 1$, $\mathfrak{L}_n(G)$ *is a free* $\mathbf{Z}$*-module of rank* a_n *given by* (A).

Now let L be a tame link in S^3 with components $K_1, \dots, K_m$ and let G be the group of the link. The linking diagram of L is the edge-weighted graph Γ whose vertices are the components of L, with two vertices K_i and K_j being joined by an edge of weight l_{ij} = the linking number of K_i with K_j. By convention, we delete from the diagram any edges of weight zero. In the above presentation of G/G_{q+1}, the relators $r_i^{(q)}$ have the form

$$r_i^{(q)} \equiv \prod_{j \neq i} [x_i, x_j]^{l_{ij}} \mod F_3.$$

Assume that for each i there is a j such that $l_{ij} \neq 0$, i.e., that each initial form ρ_i is of degree 2. Then

$$\rho_i = \sum_{j \neq i} l_{ij} [\xi_i, \xi_j].$$

The conjecture of Murasugi we are dealing with concerns the property (M).

$$\text{(M)} \qquad \mathfrak{L}_n(G) \cong \mathfrak{L}_n(F(m-1)) \text{ for } n \geq 2.$$

His conjecture was that property (M) held if $l_{ij} = \pm 1$ for all i, j. In [8] Maeda proved the following sharpened form of this conjecture: property (M) holds if Γ has a spanning subtree whose edges have weights ± 1. His proof uses the Chen groups $C_n(G) = G_n G''/G_{n+1}G''$, where $G' = [G, G]$ is the derived group of G and G'' is the second derived group of G. He also shows that property (M) is equivalent to $(\widetilde{\mathrm{M}})$:

$$(\widetilde{\mathrm{M}}) \qquad C_n(G) \cong C_n(F(m-1)) \text{ for } n \geq 2.$$

Using this, Massey and Traldi [9] were able to show that $(\widetilde{\mathrm{M}})$ was equivalent to the statement that $G_2/G_3 \cong H_2/H_3$, where $H = F(m-1)$. In this paper we give a completely independent proof of this result that avoids the use of Chern groups.

Theorem 2. *The following are equivalent*:

(1) *The linking diagram* Γ *is connected* mod p *for every prime* p;
(2) *Property* (I) *holds with* $s = m$ *and* $d_1 = \cdots = d_{s-1} = 2$;
(3) *Property* (M);
(4) *The group* $\mathfrak{L}_2(G)$ *is a free* **Z**-*module of rank* $(m-1)(m-2)/2$.

The graph Γ is said to be connected modulo p if there is a spanning subtree of Γ whose edges have weights that are not congruent to zero modulo p. This theorem sharpens Theorem 3.7 of [2].

Corollary (Massey-Traldi). *Let* G *be the group of an* m-*component link in* S^3. *Then, for all primes* p, *the rank of* $\mathfrak{L}_2(G) \otimes_{\mathbf{Z}} \mathbf{F}_p$ *is greater than or equal to* $(m-1)(m-2)/2$ *with equality for all primes* p *if and only if the equivalent conditions of Theorem* 2 *hold.*

This result follows immediately from the fact that $\mathfrak{L}_2(G)$ is the quotient of $\mathfrak{L}_2(F)$ by the subgroup generated by those ρ_i, which are of degree 2.

Proof of Theorem 1. It remains only to prove that (2) implies (1). So assume that (2) holds and let U be the enveloping algebra of $\mathfrak{L}/\mathfrak{r}$. Let V be the free U-module on generators $v_1, \ldots, v_{s-1}$ with v_i of degree d_i and let $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$. Let $\gamma \colon V \to M$ be the U-module homomorphism with $\gamma(v_i)$ equal to the image of ρ_i in M. Then (I) holds if and only if U is **Z**-free and γ is bijective.

Now let p be a prime, let $\overline{\mathfrak{L}} = \mathfrak{L} \otimes \mathbf{F}_p$, let $\overline{\rho}_i$ be the image of ρ_i in $\overline{\mathfrak{L}}$, let $\overline{\mathfrak{r}}$ be the ideal of $\overline{\mathfrak{L}}$ generated by $\overline{\rho}_1, \ldots, \overline{\rho}_{s-1}$ and let $\overline{U}$ be the enveloping algebra of $\overline{\mathfrak{L}}/\overline{\mathfrak{r}}$. If $\overline{V} = V \otimes \mathbf{F}_p$ and $\overline{M} = \overline{\mathfrak{r}}/[\overline{\mathfrak{r}}, \overline{\mathfrak{r}}]$, let $\overline{\gamma} \colon \overline{V} \to \overline{M}$ be the $\overline{U}$-module homomorphism sending $\overline{v}_i = v_i \otimes 1$ to the image of $\overline{\rho}_i$ in $\overline{M}$. Then (I) holds if and only if $\overline{\gamma}$ is bijective for every prime p.

Let $N(t)$ denote the Poincaré series of a graded vector space N. If $\overline{W}$ is the enveloping algebra of $\overline{\mathfrak{r}}$ we have $\overline{W}(t) = (1 - \overline{M}(t))^{-1}$ and, if $\overline{K}$ is the kernel of $\overline{\gamma}$ we have

$$\overline{M}(t) = (t^{d_1} + \cdots + t^{d_{s-1}})\overline{U}(t) + \overline{K}(t).$$

Using the fact that $\overline{U}(t)\overline{W}(t) = (1-mt)^{-1}$, we obtain

$$\overline{U}(t)^{-1} = 1 - mt + \sum_{1\le i<s} t^{d_i} + \overline{K}(t)\overline{U}(t)^{-1}.$$

It follows that $\overline{\gamma}$ is bijective in degrees $\le n$ if and only if

$$\text{(X)} \qquad \overline{U}(t) \equiv \left(1 - mt + \sum_{1\le i<s} t^{d_i}\right)^{-1} \mod t^{n+1}.$$

Call $\rho_1, \dots, \rho_{s-1}$ n-inert (resp. n-inert $\bmod p$) if γ (resp. $\overline{\gamma}$) is bijective in degrees $\le n$. Then, since the sequence $\rho_1, \dots, \rho_{s-1}$ is n-inert if and only if it is inert $\bmod p$ for every prime p, we see that the sequence $\rho_1, \dots, \rho_{s-1}$ is n-inert if and only if (X) holds for every prime p.

Now suppose that $\rho_1, \dots, \rho_{s-1}$ is n-inert for some n (this is true for $n = 0$). Then by the proof of [7, Theorem 1, p. 54] we obtain that $\mathfrak{L}/\mathfrak{R}$ equals $\mathfrak{L}/\mathfrak{r}$ in degrees $\le n+1$. Since (2) holds we obtain (X) with n replaced by $n+1$, i.e., that $\rho_1, \dots, \rho_{s-1}$ is $(n+1)$-inert. Thus $\rho_1, \dots, \rho_{s-1}$ is n-inert for all n, which is Property (I).

Proof of Theorem 2. The implication $(1) \Rightarrow (2)$ is a result of Anick [2, Proposition 3.5].

If (2) holds then we have shown in [7] that $\mathfrak{r} = \mathfrak{R}$ and hence that $\mathfrak{L}(G) = \mathfrak{L}/\mathfrak{r}$. By Anick [2, Theorem 1.6], or by results of [2] in conjunction with [4], the enveloping algebra of $\mathfrak{L}(G)$ is a free **Z**-module whose nth homogeneous component is of rank equal to the coefficient of t^n in the formal power series

$$(F) \qquad (1 - mt + (m-1)t^2)^{-1} = ((1-t)(1-(m-1)t))^{-1}.$$

The right-hand side of (F) is the Poincaré series for the enveloping algebra of $\mathfrak{L}(F(1)) \oplus \mathfrak{L}(F(m-1))$, hence $\mathfrak{L}(G) \cong \mathfrak{L}(F(1)) \oplus \mathfrak{L}(F(m-1))$ as graded abelian groups. Since $\mathfrak{L}_n(F(1)) = 0$ for $n \ge 2$, Property (M) follows. That (3) implies (4) is immediate.

To show that (4) implies (1) let $\Lambda = (\lambda_{i(j,k)})$ be the $m \times m(m-1)/2$ matrix, whose rows are indexed by $1, \dots, m$ and whose columns are indexed by the pairs (i, j) with $1 \le i < j \le m$, with $\lambda_{i(i,j)} = l_{ij}$, $\lambda_{j(i,j)} = -l_{ij}$ and $\lambda_{i(j,k)} = 0$ otherwise. Then, since Λ is a presentation matrix for the abelian group $\mathfrak{L}_2(G)$, we see that (4) holds if and only if for every prime p the reduction of $\Lambda \bmod p$ has rank $m-1$. Since the (i, j)th column of Λ is l_{ij} times a column vector with entries equal to zero except in the ith and jth rows where the entries are 1 and -1 respectively, we see that the linking diagram of L is connected $\bmod p$ if and only if the reduction of $\Lambda \bmod p$ has rank $m-1$.

Remarks. In [8] Maeda uses the following independence condition:

(MI) $\mathfrak{L}'/\mathfrak{r} + \mathfrak{L}''$ *is a free* **Z**-*module and the quotient* $\mathfrak{r}/\mathfrak{r} \cap \mathfrak{L}''$ *is a free* $U(\mathfrak{L}/\mathfrak{L}')$-*module on the images of* $\rho_1, \dots, \rho_{s-1}$.

Here $\mathfrak{L}' = [\mathfrak{L}, \mathfrak{L}]$ is the derived algebra of $\mathfrak{L}$ and $\mathfrak{L}''$ is the second derived algebra. His main result is to show that (MI) implies that $\mathfrak{R} = \mathfrak{r}$ and that $\mathfrak{L}(G)$ is **Z**-free. However he does not give a formula for the rank of $\mathfrak{L}_n(G)$ except in the case each ρ_i is of degree 2, which is enough to settle Murasugi's conjecture. This he does by giving an algorithm for getting a basis of $\mathfrak{L}_n(G)$ that shows that the rank in question depends only on the degrees of the ρ_i s and their multiplicities. He is therefore reduced to computing the rank in a special case, which he does in the case of degree 2.

We finish by showing that (MI) implies (I). Since the sequence

$$0 \to \mathfrak{r}/\mathfrak{r} \cap \mathfrak{L}'' + \mathfrak{L}'/\mathfrak{L}'' \to \mathfrak{L}'/\mathfrak{r} + \mathfrak{L}'' \to 0$$

is exact, we see that (MI) holds if and only if the elements

$$(*) \qquad \mathrm{ad}(\xi_1)^{i_1} \cdots \mathrm{ad}(\xi_m)^{i_m}(\rho_j), \qquad i_k \geq 0,\ 1 \leq j \leq s-1,$$

are part of a basis for the graded free **Z**-module $\mathfrak{L}'/\mathfrak{L}''$ and hence part of a basis of the free Lie algebra $\mathfrak{L}'$, cf. [6, Prop. 2]. The elements $(*)$ generate $\mathfrak{r}$ as an ideal of $\mathfrak{L}'$ and hence $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free $U(\mathfrak{L}'/\mathfrak{r})$-module on their images in $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$, cf. [3, Prop. 10]. Since $U(\mathfrak{L}/\mathfrak{L}')$ is the polynomial ring over **Z** on $\xi_1, \dots, \xi_m$ it follows that $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free $U(\mathfrak{L}/\mathfrak{r})$-module on the images of $\rho_1, \dots, \rho_{s-1}$. Also, $\mathfrak{L}/\mathfrak{r}$ is **Z**-free since $\mathfrak{L}/\mathfrak{L}'$ is **Z**-free and $\mathfrak{L}'/\mathfrak{r}$ is a free Lie algebra over **Z**. Hence (I) holds.

From the above proof one also deduces that (MI) implies that the derived algebra of $\mathfrak{L}(G)$ is a free Lie algebra over **Z**.

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