

GROWTH OF TWO-STEP SOLVABLE LIE ALGEBRAS

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ABSTRACT. It is proved that every finitely generated infinite-dimensional two-step solvable Lie algebra has polynomially bounded growth. As a corollary, it is shown that the universal enveloping algebras of such Lie algebras are not Noetherian but of Krull domain.

1. INTRODUCTION

Let A be an algebra (not necessarily associative) over the field k . (Throughout this note, we assume that k is of characteristic zero.) When A is generated by the finite set X , we use X^n to denote the subspace of A spanned by all products of n or fewer elements of X . We define the growth function $\gamma(n; X)$ of A with respect to X as $\gamma(n; X) = \dim_k X^n$. We say that A has exponential growth if there exist positive constants K , $a > 1$ and an integer b such that $a^n \leq K\gamma(bn; X)$. Otherwise, A has subexponential growth. We also say that A has polynomially bounded growth if there exist a positive constant K and integers d and b such that $\gamma(n; X) \leq K(bn)^d$. Of course, these properties of A do not depend on the choice of generating set X .

In particular, when A has polynomially bounded growth, we set $Gk \dim(A) = \limsup(\log \gamma(n; X) / \log n) = \inf\{d \in \mathbb{R} \mid \gamma(n; X) \leq K(bn)^d \text{ for almost every } n\}$. This number also does not depend on the choice of X , and is a very important invariant of A .

The behavior of the growth function has various influences on the structure of A . (For a general reference, see [7].) Our interest is mainly in infinite-dimensional Lie algebras and their enveloping algebras. In this field, we know that simple graded Lie algebras such as affine Kac-Moody algebras, the Virasoro algebra, and Cartan algebras have polynomially bounded growth (see [5]).

In this note, we consider the class of finitely generated infinite-dimensional two-step solvable Lie algebras. Our main result asserts that every Lie algebra of this class has polynomially bounded growth (Theorem 2.1). This implies that the universal enveloping algebra of a Lie algebra of this class is not Noetherian but is of Krull domain (Remark 3.1 and Theorem 3.1).

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2. MAIN RESULT

Let g be a (generally infinite-dimensional) Lie algebra over k . The derived series associated with g , which is the sequence of ideals of g , is defined successively as follows:

$$D_0g = g, D_{k+1}g = [D_kg, D_kg].$$

A Lie algebra g is called solvable if there is a positive integer N such that $D_Ng = 0$. Moreover, if $D_{N-1}g \neq 0$ and $D_Ng = 0$, then g is called N -step solvable.

Let S_N denote the class of finitely generated, infinite-dimensional N -step solvable Lie algebras over k , and let S denote the union $\cup_N S_N$. We then set p_k the natural projection from D_kg to $D_kg/D_{k-1}g$.

Lemma 2.1. *If g is a Lie algebra of S , then g/D_1g is finite-dimensional. In particular, D_1g is the infinite-dimensional commutative ideal of g for $g \in S_2$.*

Proof. Let X be a finite generating subspace of g . Since $[X, X] \subset D_1g$, $X^n \subset D_1g$ and $g = \cup X^n \subset X + D_1g$, $p_0(g) = p_0(X)$. This implies the finite-dimensionality of g/D_1g . The other statement follows from this.

Remark 2.1. It is well known that every finitely generated nilpotent Lie algebra is of finite dimension (see [3]). Hence S_1 is empty and S contains no nilpotent Lie algebra.

Due to Lemma 2.1, we know that $\dim(g/D_1g)$ is finite for each g of S_N . Hence we can define the subclass $S_{N,d}$ of S_N as the set $\{g \in S_N \mid \dim(g/D_1g) = d\}$. It is easy to see that $d \geq 2$ and $\cup_d S_{N,d} = S$.

The following linear algebraic and combinatorial argument is the key to this section.

Lemma 2.2. *Let V be a vector space over k and D be a finite-dimensional subspace of $\text{End}(V)$. Assume that D consists of pairwise commutative endomorphisms. Then for any finite-dimensional subspace W of V , we can choose a positive integer N and constants K and K' such that the following inequality holds:*

$$K \cdot n \leq \dim(D^n W) \leq K' \cdot \dim W \cdot n^{\dim D} \quad \text{for all } n \geq N.$$

Proof. The first part of the inequality follows from the fact that g is finitely generated and of infinite dimension.

Next, let d_1, d_2, \dots, d_L be a basis of D , where L denotes the dimension of D . Then, due to the commutativity of D , we get:

$$(2.1) \quad D^n W = \sum_{k^{(j)} \geq 0, k^{(1)} + \dots + k^{(L)} \leq n} (d_1)^{k^{(1)}} (d_2)^{k^{(2)}} \dots (d_L)^{k^{(L)}} W.$$

Since

$$(2.2) \quad \dim((d_1)^{k^{(1)}} (d_2)^{k^{(2)}} \dots (d_L)^{k^{(L)}} W) \leq \dim W,$$

we get:

$$(2.3) \quad \dim(D^n W) \leq \sum_{k(j) \geq 0, k(1) + \dots + k(L) \leq n} \dim W.$$

Now we set $\pi(n, L) = \#\{(k(1), \dots, k(L)) \mid k(i) \geq 0, k(1) + \dots + k(L) = n\}$, where “#” means the cardinal of a set. Then we can write:

$$(2.4) \quad \dim(D^n W) \leq \dim W \cdot \left(\sum_{0 \leq k \leq n} \pi(k, L) \right)$$

Now we can easily show that $\pi(n, L) = \binom{n+L-1}{L-1}$ (a binomial coefficient). This formula and (2.4) give the desired inequality (c.f. Lemma 1.5 of [5]).

Theorem 2.1. *Let g be a Lie algebra of $S_{2,d}$. Then g has polynomially bounded growth and $1 \leq GK \dim(g) \leq d$.*

Proof. Let D be a subspace of g complementary to $D_1(g)$. We know that D has the finite dimension d . Since g is finitely generated, we can choose a vector subspace W of $D_1(g)$ such that $D \oplus W$ generate g as Lie algebra. Now $D_1(g)$ is the commutative Lie algebra (Lemma 2.1) and $\text{ad}(D)|_{D_1(g)}$ is the pairwise commutative family of endomorphisms of $D_1(g)$.

Since $[D, W] \subset D_1(g)$ and $[W, W] = 0$, it is easy to show that $(D \oplus W)^n = D \oplus \sum_{k \leq n} \text{ad}(D)^k \cdot W$. Hence Lemma 2.2 implies the first part of the theorem.

Remark 2.2. (1) The following example gives a Lie algebra g of $S_{2,d}$ for $d \geq 2$ satisfying $GK \dim(g) = 1$; $g = kx_1 \oplus kx_2 \oplus \dots \oplus kx_{d-1} \oplus ky_1 \oplus ky_2 \oplus ky_3 \oplus \dots$ (with bracket relations $[x_i, x_j] = [y_i, y_j] = 0$ and $[x_i, y_j] = y_{j+1}$).

This means that the best possibility to estimate lower bounds of GK -dimensions is in Theorem 2.1. For upper bounds, see problem I of §4.

Note that the example of Smith [8] corresponds to the case of $d = 2$. So we may say that Theorem 2.1 is a generalization of the result of [8].

(2) Let h be a Lie algebra of S_N and $U(h)$ be the universal enveloping algebra of h . Using the usual action of h on $U(h)$ (that is, left multiplication), we can construct the semidirect product $g = h \oplus U(h)$ such that $U(h)$ is a commutative ideal of g . Then g belongs to S_{N+1} and does not have polynomially bounded growth.

Hence, it is found that there exists a Lie algebra of S_N which does not have polynomially bounded growth for each $N \geq 3$ (cf. (2) of Remark 3.1).

3. ENVELOPING ALGEBRAS

In this section, we discuss algebraic properties of universal enveloping algebras of Lie algebras of S . We write $U(g)$ as the universal enveloping algebra of a Lie algebra g .

Recall that an associative k -algebra A is called an Ore domain if $xA \cap yA \neq 0$ and $Ax \cap Ay \neq 0$ for every nonzero elements $x, y \in A$. If A is an Ore domain, then A has the quotient division ring $Q(A)$. A is of maximal order provided

that there exists no ring B with $A \subsetneq B \subseteq Q(A)$ such that $aBb \subseteq A$ for some nonzero element $a, b \in A$.

A Krull domain in the sense of Chamarie [2] is an Ore domain A which is of maximal order with ascending chain condition on closed right (resp. left) ideals, where a right ideal I of A is called a closed right ideal provided that $I = \{x \in A \mid xJ \subset I \text{ for some right ideal } J \text{ of } A \text{ such that } J^* = \text{Hom}_A(J, A) = A\}$.

As a corollary of Theorem 2.1, we get the following result:

Theorem 3.1. *Let g be a Lie algebra of S_2 . Then the universal enveloping algebra $U(g)$ of g is of Krull domain.*

Proof. It is well known that $U(g)$ has no zero divisor. Hence, due to Lemma 2.1 and the results of [6], [2], and [4], it is sufficient to show that $U(g)$ does not contain a two-generator free algebra. (See also Chapter 4 of [7].)

It is clear that if $U(g)$ contains a two-generator free algebra, then $U(g)$ must have exponential growth. On the other hand, Theorem 2.1 asserts that g has subexponential growth. Now, [8] showed that if a Lie algebra has subexponential growth, then its universal enveloping algebra also has subexponential growth. Hence $U(g)$ has subexponential growth. This completes the proof.

Remark 3.1. (1) It is known [1] that every universal enveloping algebra of infinite-dimensional solvable Lie algebra is not Noetherian. Hence Theorem 3.1 asserts that universal enveloping algebras of finitely generated infinite dimensional 2-step solvable Lie algebras give examples of finitely generated algebras not of Noetherian but of Krull domain.

(2) Note that (2) of Remark 2.2 does not necessarily imply that the universal enveloping algebra of a general solvable Lie algebra must have exponential growth. In fact, the same discussion as in the proof of Theorem 3.1 implies that if h has subexponential growth then so does g (c.f. Problem II of §4).

4. PROBLEMS

Finally, we propose some problems.

- I. Concerning (2) of Remark 2.2: Is the estimate of upper bounds of GK-dimensions in Theorem 2.1 the best possible?
- II. Concerning (2) of Remark 3.1: Does a finitely generated solvable Lie algebra g have subexponential growth? In particular, is the universal enveloping algebra of g Krull?

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